International Journal of Solids and Structures 51 (2014) 2529–2538

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Transverse conductivity and longitudinal shear of elliptic fiber composite with imperfect interface

V.I. Kushch *, V.S. Chernobai

Institute for Superhard Materials of the National Academy of Sciences, 04074 Kiev, Ukraine

ARTICLE INFO

Article history:
Received 28 December 2013
Received in revised form 20 February 2014
Available online 28 March 2014

Keywords:
Composite
Ellipse
Imperfect interface
Effective conductivity
Multipole expansion
Unit cell model

ABSTRACT

The paper addresses the problem of calculating the local fields and effective transport properties and longitudinal shear stiffness of elliptic fiber composite with imperfect interface. The Rayleigh type representative unit cell approach has been used. The micro geometry of composite is modeled by a periodic structure with a unit cell containing multiple elliptic inclusions. The developed method combines the superposition principle, the technique of complex potentials and certain new results in the theory of special functions. An appropriate choice of the potentials provides reducing the boundary-value problem to an ordinary, well-posed set of linear algebraic equations. The exact finite form expression of the effective stiffness tensor has been obtained by analytical averaging the local gradient and flux fields. The convergence of solution has been verified and the parametric study of the model has been performed. The obtained accurate, statistically meaningful results illustrate a substantial effect of imperfect interface on the effective behavior of composite.

1. Introduction

The interfaces play an important, often dominant role in determining the local and overall behavior of heterogeneous solids. The "perfect interface bonding" (more correctly, perfect thermal/mechanical contact) assumption widely used in micromechanics is merely a more or less appropriate idealization. In fact, the interfaces are *always* imperfect: the atomic lattices mismatch, poor mechanical or chemical adherence, surface contamination, oxide and interphase diffusion/reaction layers, coatings, interface debonding or cracking, etc. are possible reasons of imperfectness. Even in the case of ideal contact between the constituents, the interface resistance due to interfacial phonon scattering makes the composite properties size-dependent (e.g., Every et al., 1992). Nanocomposites (see, e.g., Luo and Wang, 2009 and the references therein) give an another example of the size dependence due to interface effects.

In this paper, we focus on the two-dimensional (2D) scalar (conductivity and out-of-plane shear) models of matrix type composites. The problems involving composites of circular inclusions with imperfect interface have received a considerable attention in the literature. Probably, the most known is the work by Hasselman and Johnson (1987) which extends the famous Maxwell's formula for effective conductivity to a fibrous composite with imperfect interface. An effect of interfacial characteristics on the effective thermal conductivity of isotropic two-dimensional periodic (square or hexagonal) composites of circular cylinders is studied by Lu and Lin (1995). An approximate solution for a random composite of imperfectly bonded fibers (Lu and Song, 1996) takes into account pair wise fiber-to-fiber interactions and radial distribution function. Graham and McDowell (2003) estimated thermal conductivity of random fiber composite with imperfect interface by the finite element analysis of the many-inclusion cell model. These and other similar (Achenbach and Zhu, 1989; Hashin, 1990; Gao, 1995 among others) works are based on the assumption that flux/traction is continuous whereas temperature/displacement is discontinuous across the interface. Specifically, a jump in the displacement is proportional, in terms of "spring-factor-type" interface parameters, to the interface traction. When these interface parameters uniform along the entire length of the material interface, the model is said to represent a *homogeneously* imperfect interface. The case of *inhomogeneous* interface in out-of-plane shear was considered by Ru and Schiavone (1997).

In the real-life composites, we do not expect the inclusions to be of exact canonical shape. The above mentioned models are adequate for the heterogeneous solids with equiaxial inclusions where the mean radius is the only length parameter of inclusion. When the inclusion's shape deviates considerably from the circular one, we need an additional length parameter to quantify it. In this case, an ellipse (also possessing two length parameters) appears to be

* Corresponding author. Tel./fax: +380 444329544.
E-mail address: vkushch@bigmir.net (V.I. Kushch).

http://dx.doi.org/10.1016/j.ijsolstr.2014.03.024
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more appropriate model shape. In particular, an infinitely thin elliptic hole is a convenient model of the straight crack. Therefore, the "solid with elliptic inclusions" model seems appealing in both theoretical and practical aspects.

The solution for a single elliptic inclusion is well known (see, e.g., Ru and Schiavone, 1996, and the references therein). At the same time, the analytical solutions for the interacting elliptic inclusions are limited to a few. A certain progress is observed in the conductivity problem where the multipole expansion solution has been obtained for the finite (Yardley et al., 1999) and periodic (Yardley et al., 2001; Kuo, 2010) arrays of elliptical cylinders. The complete solutions for a finite array of elliptic inclusions in the plane (Kushch et al., 2005) and half-plane (Kushch et al., 2006) have been obtained by combining the multipole expansion approach with the Kolosov–Muskhelishvili's technique of complex potentials. This approach has been further developed and applied to evaluation of the stress intensity factors (Kushch et al., 2009a) and effective stiffness (Kushch et al., 2009b) of cracked solids. The theory of the multipole expansion method in application to the solids with elliptic inclusions is given in the book by Kushch (2013). Among the available numerical studies on effective conductivity of fiber composites, we mention the work by Lu (1994) who applied the boundary collocation scheme for evaluating effective conductivity of the rectangular arrays of elliptic inclusions. Byström (2003) studied the many-particle cell model for the periodic and random structure composites with circular or elliptic inclusions by the finite element method.

To our best knowledge, the elliptic fiber composites with imperfect interface never been addressed in the micromechanics literature. The paper by Shen et al. (2000) who considered a single elliptic inclusion with homogeneously imperfect interface is probably the only effort in this direction. The aim of our work is to close this gap by developing the micromechanical model of elliptic fiber composite for the conductivity and out-of-plane shear problems, able to take into account microstructure of composite and interactions between the inclusions with imperfect bonding to the matrix.

The outline of this paper is as follows. First, we formulate the problem in terms of complex potentials. Second, a general solution of the problem for a single, imperfectly bonded elliptic inclusion in the inhomogeneous far field is derived and tested numerically. Next, we incorporate this solution into a general scheme of the multipole expansion method to get a complete solution for a representative unit cell (RUC) model of elliptic fiber composite with imperfect interface. By analytical averaging the local gradient and flux fields, the exact finite form expression of the effective conductivity tensor has been obtained. The effect of interface conductivity on the effective behavior of composite has been evaluated. The background theory is provided in Appendices.

2. Governing equations in terms of complex variables

We consider a steady heat conduction in the unidirectional elliptic fiber composite due to transverse heat flux. In this case, the two-dimensional (2D) model is adequate to study the phenomenon. We assume both the matrix and fibers to be isotropic. The governing equation is $\nabla \cdot \mathbf{q} = 0$, where $\mathbf{q} = -\lambda \nabla T$ is the heat flux vector. Also, λ is the thermal conductivity, T and ∇T is the temperature and its gradient, respectively. In the case of constant λ, T obeys Laplace equation $\nabla^2 T = 0$.

Our analysis employs the technique of complex potentials (e.g., Muskhelishvili, 1953). In terms of the complex variable $z = x_1 + \mathrm{i}x_2$ representing the position vector $\mathbf{x} = (x_1, x_2)^T$ in Ox_1x_2 plane, Laplace equation reduces to

$$\nabla^2 T = \frac{\partial^2 T}{\partial z \partial \bar{z}} = 0.$$

The temperature T and complex heat flux $q = q_1 + \mathrm{i}q_2$ are expressed in terms of the complex potential $\varphi(z)$ as

$$T = \mathrm{Re}\,\varphi(z); \quad q = q_1 + \mathrm{i}q_2 = -\lambda \overline{\varphi'(z)}. \tag{1}$$

Here and below, prime denotes differentiation with respect to the whole argument and over-bar denotes the complex conjugate.

The mathematically equivalent mechanical problem in the 2D elasticity theory is out-of-plane shear, where u_3 is the only non-zero component of displacement vector $\mathbf{u}$:

$$u_1 = u_2 = 0; \quad u_3 = w(x_1, x_2).$$

In this case, two non-zero components of the stress tensor are σ_{13} and σ_{23}. The stress equilibrium equation $\nabla \cdot \sigma = 0$ takes the form

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0; \tag{2}$$

the Hooke's law reduces to

$$\sigma_{i3} = 2\mu\varepsilon_{i3} = \mu \partial w / \partial x_i, \quad i = 1, 2. \tag{3}$$

It follows from Eqs. (2) and (3) that $\nabla^2 w = 0$ whereas the strain compatibility condition

$$\frac{\partial \varepsilon_{13}}{\partial x_2} = \frac{\partial \varepsilon_{23}}{\partial x_1} = \frac{1}{2} \frac{\partial^2 w}{\partial x_1 \partial x_2}$$

is obeyed identically. This problem is readily reformulated in terms of the complex potentials (Muskhelishvili, 1953). For $w = \mathrm{Re}\,\varphi(z)$, Eq. (3) takes the form analogous to Eq. (1): namely, the complex stress $\sigma = \sigma_{13} + \mathrm{i}\sigma_{23} = \mu \overline{\varphi'(z)}$. Hence, the conductivity problem we consider below can be also interpreted in the mechanical context as the out-of-plane shear of elastic fibrous composite, with replace T to w, $(-\lambda)$ to μ and q to σ.

3. Single inclusion in an inhomogeneous far field

3.1. The problem statement

Consider an unbounded plane, or matrix, containing a single elliptic inclusion. All the matrix- and inclusion-related quantities are indexed by "0" and "1", respectively: $T = T^{(0)}$ and $\lambda = \lambda_0$ in the matrix, $T = T^{(1)}$ and $\lambda = \lambda_1$ in the inclusion. To describe geometry of the problem, we introduce the Cartesian coordinate frame Ox_1x_2 so that its origin coincides with the centroid of ellipse whereas the Ox_1 and Ox_2 axes are directed along the major and minor axes of the ellipse. An aspect ratio of the ellipse is $e = l_2/l_1$, where l_1 and l_2 are the major and minor, respectively, semi-axes of the ellipse; its area $S_1 = \pi l_1 l_2$. Another derivative geometric parameter to be used in our analysis is the inter-foci distance $2d$, where $d = \sqrt{l_1^2 - l_2^2}$.

Alongside with conventional complex variable $z = x_1 + \mathrm{i}x_2$, we will use the "elliptic" complex variable $\xi = \zeta + \mathrm{i}\eta$ introduced (e.g., Sneddon and Berry, 1958) as

$$z = d \cosh \xi = \frac{d}{2}(\upsilon + \upsilon^{-1}), \quad \upsilon = \exp \xi. \tag{4}$$

In fact, Eq. (4) defines an elliptic coordinate frame with ζ and η as "radial" and "angular" coordinates, respectively. In particular, the coordinate curve $\zeta = \zeta_0$ specified by the condition

$$\zeta_0 = \ln\left(\frac{l_1 + l_2}{d}\right) = \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right) \tag{5}$$

coincides with the boundary of elliptic inclusion. Also, we denote $\upsilon_0 = \exp \zeta_0$. It is important that at this boundary the functions $\upsilon^k = \upsilon_0^k \exp \mathrm{i}k\eta$ depend only on the angular coordinate η. This fact makes the complex variable ξ particularly useful for the domains