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Dynamic stability of externally pressurized elastic rings subjected to high rates of loading

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ABSTRACT

Of interest here is the influence of loading rate on the stability of structures where inertia is taken into account, with particular attention to the comparison between static and dynamic buckling. This work shows the importance of studying stability via perturbations of the initial conditions, since a finite velocity governs the propagation of disturbances. The method of modal analysis that determines the fastest growing wavelength, currently used in the literature to analyze dynamic stability problems, is meaningful only for cases where the velocity of the perfect structure is significantly lower than the associated wave propagation speeds.

As a model structure to illustrate this point we select an elastic ring subjected to external hydrostatic pressure which is applied at different rates ϵ (appropriately non-dimensionalized with respect to elastic axial wave speed). The ring's stability is studied by following the evolution of a localized small perturbation. It is shown that for small values of the applied loading rate, the structure fails through a global mode, while for large values of the applied loading rate the structure fails by a localized mode of deformation. An analytically obtained localization time t_l is found to be a very good estimate of the onset of instability time at high loading rates.

1. Introduction

The issue of dynamic stability of structures is an important engineering problem and as such has drawn considerable attention. The first investigation in this area appears to be the work of Koning and Taub (1933), who investigated the influence of inertia in a simply supported imperfect column subjected to a sudden axial load. A substantial amount of work followed that investigated the response of, mainly elastic, structures to impulse or time-dependent loads. As a result, and due to the many possible definitions for the stability of time-dependent systems, the term *dynamic stability* encompasses many classes of problems and different physical phenomena and has many interpretations, with inertia being the only common denominator.

In the absence of inertia, the processes of failure by a bifurcation instability mode in elastic solids and structures is well understood (e.g. Brush and Almroth, 1975) and a general asymptotic analysis, termed Lyapunov–Scmidt–Koiter (LSK), has been developed for their study. The first effort to use the LSK general analysis for the dynamic stability problem of an elastic structure appears to be Budiansky and Hutchinson (1964), where the authors proposed an asymptotic analysis of the time-dependent problem using the eigenmodes of the static problem. Alternative methods, based on upper and lower bounds of the structure's energy have also been proposed and the interested reader is referred to Chapter 12 in Simitses and Hodges (2006) for a well written account of this approach.

Another idea, popular in fluid mechanics, has also been adopted for the dynamic stability analysis of solids with more general constitutive laws under high rates of loading, according to which one seeks the solid's fastest growing eigenmode. This type of analysis is also termed the *method of frozen coefficients*, since the resulting PDE system of the linearized stability equations become autonomous by virtue of ignoring the time-independence of their coefficients. This method has been repeatedly applied in the study of dynamic stability of elastoplastic bars and rings under high loading rates where the size of fragments is of interest (e.g. see Shenoy and Freund, 1999; Sorensen and Freund, 2000; Mercier and Molinari, 2003). However, recent experimental evidence from rapidly expanding electromagnetically loaded metallic rings by Zhang and Ravi-Chandar (2006, 2008) finds no evidence of a dominant wavelength at the necked pattern of the rings. As explained by

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these authors, using the fastest growing eigenmode to predict the onset of failure is physically meaningful provided that the loading rate is much slower than the speed of propagation of perturbations in the solid or structure at hand. For high loading rates, commensurate with some characteristic wave propagation speed in the structure, a novel approach to the stability analysis is required, namely the study of evolution of localized perturbations.

In contrast to the above mentioned cases of structures under rapid extension, of particular interest in this work is the influence of loading rate on the stability of structures under compression that exhibit an instability even under quasistatic loading. As a model structure to illustrate these ideas, we select an elastic ring subjected to external hydrostatic pressure which is applied at different rates ϵ (appropriately non-dimensionalized with respect to elastic axial wave speed). Of course such a classical topic has been treated repeatedly in the mechanics literature; following the work of Carrier (1945), different linear and nonlinear versions of the ring dynamical equations of increasing complexity have been proposed (e.g. Morley, 1961; Goodier and McIvor, 1964; Boresi and Reichenbach, 1967; Wah, 1970; Graff, 1971; Simmonds, 1979; Dempsey, 1996) to study their vibrations. The stability of rings subjected to impulsive or step loadings has also been repeatedly studied (e.g. Goodier and McIvor, 1964; Lindberg, 1964; Florence, 1968; Anderson and Lindberg, 1968a; Lindberg, 1974; Simmonds, 1979; Lindberg and Florence, 1987; Amabili and Paidoussis, 2003). These studies rely on modal analysis using Fourier series whose truncation leads nonlinear amplitude equations and showed that dynamic buckling is triggered by flexural modes. At leading order, the dynamics of flexural modes are governed by Mathieu-Hill equations whose characteristic curves of associated Mathieu functions delineate boundaries of instability domains within the control parameter plane of load versus ring's slenderness. For an account of dynamic stability problems in rings, the interested reader is referred to the book Graff (1975) and references quoted therein.

All the above-mentioned works were concerned with the stability of ring vibrations and not with their stability at high loading rates as is the case of interest here. Our investigation is further motivated by work involving rings high strain-rate using electromagnetic loading – since this method avoids propagating waves – under tension (Gourdin, 1989; Triantafyllidis and Waldenmyer, 2004; Zhang and Ravi-Chandar (2006); Zhang and Ravi-Chandar (2008)) that study the influence of high loading rate on metal ductility and in particular by experiments in ring and cylinder under electromagnetic compression by Anderson and Lindberg (1968b) and Jones and Okawa (1976), since these experiments combine structural instability with rapid loading. It is the most recent experimental work of Mainy (2012) that serves as the starting point for this investigation, and in particular the localized failure patterns observed (see Fig. 1), which are in marked contrast with global buckling modes of externally pressurized rings under quasistatic loading rates. In order to keep essential features such as buckling under static loading and finite wave speeds for all wavenumbers, we concentrate on the dynamics of an elastic ring following a von Karman – Timoshenko theory allowing for small strains, moderate rotations, transverse shear and rotational inertia. The ring's stability is studied by following the evolution of a localized small perturbation. It is shown that for small values of the applied loading rate the structure fails through a global mode, while for large values of the applied loading rate the structure fails by a localized mode of deformation. Following Section 1 the presentation of the work continues with Section 2, where we derive the equations of motion and outline the numerical scheme for the solution of these equations. The results are given in Section 3 where we present the linearized analysis of the initial growth/decay of a perturbation followed by numerical calculations of the evolution of a spatially localized displacement perturbation and a discussion in Section 4 concludes this work.

2. Formulation

In the first subsection we derive the equations of motion from Hamilton's variational principle, from which we deduce the structure's Euler–Lagrange equations. The numerical scheme for the solution of these equations is outlined in the second subsection.

2.1. Equations of motion

We consider a homogeneous linear elastic ring of rectangular section with thickness h, width a and cross sectional area $A = h \times a$. The ring has a mid-line radius r and follows small strain – moderate rotation Timoshenko kinematics described by $\tilde{v}(\theta), \tilde{w}(\theta), \tilde{\psi}(\theta)$ respectively the tangential and normal displacements of the ring's reference mid-line at point θ and the rotation of the section perpendicular to the mid-line, initially at θ (see Fig. 1).

To find the system's Lagrangian, we need to determine its potential and kinetic energies $\mathcal{P}$ and $\mathcal{K}$ respectively. The potential energy $\mathcal{P}$ consists of two parts: the stored elastic energy $\mathcal{P}_{\text{int}}$ plus $\mathcal{P}_{\text{ext}}$ the work potential of the externally applied uniform pressure $\tilde{\lambda}$, namely

$$\mathcal{P} = \mathcal{P}_{\text{int}} + \mathcal{P}_{\text{ext}}. \tag{1}$$

The stored elastic energy $\mathcal{P}_{\text{int}}$ is

$$\mathcal{P}_{\text{int}} = \int_0^{2\pi} \left\{ \int_{-h/2}^{h/2} \left(E\varepsilon_{\theta\theta}^2 + G\gamma_{r\theta}^2 \right) dz \right\} a\, r\, d\theta, \tag{2}$$

where the axial and shear strains $\varepsilon_{\theta\theta}$ and $\gamma_{r\theta}$ are given by

$$\varepsilon_{\theta\theta} = \frac{\tilde{v}' + \tilde{w}}{r} + \frac{1}{2}\left(\frac{\tilde{v} - \tilde{w}'}{r}\right)^2 + z\frac{\tilde{\psi}'}{r}, \quad \gamma_{r\theta} = \frac{\tilde{v} - \tilde{w}'}{r} - \tilde{\psi}, \tag{3}$$

with $f' \equiv \partial f / \partial\theta$ denoting the θ-derivative of the corresponding function and E, G the material's Young and shear moduli, respectively.

The kinematic and stress state assumptions leading to (2) and (3) are that cross-sections perpendicular to the initial middle line deform as planes, the ring is in the state of an approximate uniaxial stress $\sigma_{\theta\theta} = E\varepsilon_{\theta\theta}$, strains are small but rotations are moderate and that shear stress $\sigma_{r\theta} = G\gamma_{r\theta}$ although negligible compared to $\sigma_{\theta\theta}$ does contribute to the ring's elastic energy.

By inserting (3) into (2) and integrating through the thickness, the following expression is obtained for the internal energy

$$\mathcal{P}_{\text{int}} = \frac{1}{2}\int_0^{2\pi} \left\{ EA\left[\frac{\tilde{v}' + \tilde{w}}{r} + \frac{1}{2}\left(\frac{\tilde{v} - \tilde{w}'}{r}\right)^2\right]^2 + EI\left(\frac{\tilde{\psi}'}{r}\right)^2 + \chi G\left(\frac{\tilde{v} - \tilde{w}'}{r} - \tilde{\psi}\right)^2 \right\} r d\theta, \tag{4}$$

where $I = ah^3/12$ is the cross sectional moment of inertia and χ the shear correction factor ($\chi = 2/3$ for a rectangular section, the energy expression in (4) being valid for arbitrary cross sections with appropriate A and I expressions). For the thin rings of interest here, it is tacitly assumed that the radius of the mid-line $r \gg h$.

The work potential $\mathcal{P}_{\text{ext}}$ of the external pressure loading $\tilde{\lambda}$ applied on the ring equals $\tilde{\lambda}\Delta S$ where ΔS is the change of area due to deformation $(\tilde{v}, \tilde{w})$ enclosed by the ring's mid-line, which is given by e.g. Brush and Almroth (1975)

$$\mathcal{P}_{\text{ext}} = \tilde{\lambda} \int_0^{2\pi} \left[\tilde{w} + \frac{1}{2r}\left(\tilde{v}^2 - \tilde{v}\tilde{w}' + \tilde{v}'\tilde{w} + \tilde{w}^2\right) \right] r d\theta, \tag{5}$$

where $\tilde{\lambda}$ is taken positive when acting inwards (resulting in compressive hoop stresses $\sigma_{\theta\theta} < 0$) in the ring.