International Journal of Solids and Structures 51 (2014) 164–178

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Fundamental solutions to contact problems of a magneto-electro-elastic half-space indented by a semi-infinite punch

X.-Y. Li [a,*], R.-F. Zheng [a], W.-Q. Chen [b]

[a] School of Mechanics and Engineering, Southwest Jiaotong University, Chengdu 610031, PR China
[b] Department of Engineering Mechanics, Zhejiang University, Hangzhou 310027, PR China

ARTICLE INFO

Article history:
Received 7 May 2013
Received in revised form 6 September 2013
Available online 2 October 2013

Keywords:
Magneto-electro-elastic material
Transverse isotropy
Contact problem
Fundamental solution
Generalized potential theory method

ABSTRACT

This paper presents the fundamental contact solutions of a magneto-electro-elastic half-space indented by a smooth and rigid half-infinite punch. The material is assumed to be transversely isotropic with the symmetric axis perpendicular to the surface of the half-space. Based on the general solutions, the generalized method of potential theory is adopted to solve the boundary value problems. The involved potentials are properly assumed and the corresponding boundary integral equations are solved by using the results in literature. Complete and exact fundamental solutions are derived case by case, in terms of elementary functions *for the first time*. The obtained solutions are of significance to boundary element analysis, and an important role in determining the physical properties of materials by indentation technique can be expected to play.

1. Introduction

The fundamental solutions or Green's functions play an important role in theoretical and applied studies on the physics of solids. They can be used to solve the boundary value problems frequently encountered in the science and technology (Stakgold, 1998; Duffy, 2001), and to construct three dimensional (3D) analyses by the boundary element methods for crack, contact, defect and inclusion problems.

In the framework of elasticity, there have been some classical fundamental solutions, for instance, the Kelvin solution for an infinite isotropic body subjected to a concentrated force, and Mindlin solution for a half-infinite isotropic space. In recent several decades, a great deal of effort has been made to pursue the Green's functions for the half-infinite/infinite bodies with anisotropy and/or multi-phase coupled property (Pan and Chou, 1976, 1979a,b; Benvensite, 1992; Ding et al., 1997a,b; Pan and Han, 2004; Yang and Pan, 2004; to name a few). In particular, Ding and Jiang (2003) and Hou et al. (2005) developed the fundamental solutions for the magneto-electro-elastic (MEE) half-space with transverse isotropy, in terms of the elementary functions by the trial-and-error method. These solutions are adopted by Hou et al. (2003) and Chen et al. (2010) to study the elliptical Hertzian contact problem and to develop the general theory of indentation for the flat ended, conical and spherical punches, respectively.

The corresponding 3D exact solutions of contact problems are useful to indentation techniques, which have been widely used to characterize the physical properties of advanced materials. This has been illustrated by the pioneer works (Sneddon, 1965; Gladwell, 1980; Jonson, 1985) for isotropic elastic bodies. It was further proven by Kalinin et al. (2004, 2007) that the exact 3D contact solutions are helpful to interpreting quantitatively the response of the various scanning probe microscopies (also see Chen et al., 2010) for magneto-electro-elastic composites. In this sense, the corresponding 3D solutions within the framework of magneto-electro-elasticity are of significance, since the magneto-electro-elastic (or multiferroic) materials composites have potential applications in the intelligent systems in various engineerings, due to the strong coupling effect between the mechanical, electric and magnetic phases (Dong et al., 2004a,b; Zheng et al., 2004; Eerenstein et al., 2006; Ramesh and Spaldin, 2007; Zhai et al., 2007). To this end, Chen et al. (2010) developed the general theory of indentation over a multiferroic composite half-space by three common indenters (flat-ended, conical, and spherical punches). The half-infinite indenter, to which the corresponding problem is non-axisymmetric, has not been addressed in Chen et al. (2010).

The half-space punched by a semi-infinite indenter has been studied to some extent. Rubio-Gonzalez (2001) made a two-dimensional elasto-dynamic analysis for orthotropic materials using the Laplace and Fourier transforms conjugated with the Wiener–Hopf technique. Based on the elastostatic general solution, Fabrikant and Karapetian (1994) presented the elementary exact solutions to the corresponding mixed boundary-value problem by the potential theory method. The same method was then

* Corresponding author. Tel.: +86 28 8763 4181; fax: +86 28 8760 0797.
E-mail address: zjuparis6@hotmail.com (X.-Y. Li).

0020-7683/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijsolstr.2013.09.020

extended by Huang et al. (2007) to the contact problem in electro-elasticity. Huang et al. (2007) pointed out that the solutions for the half-infinite indenter can work as an excellent approximation to the physical variables in the half space punched by a large indenter with lengthly straight edge, which is widely employed in micro-electro-mechanical system (Waldner, 2008). In fact, this is numerically evidenced by Rubio-Gonzalez (2001). However, there is no report yet on indentation over an MEE half-space induced by the half-infinite punch, to the best of authors' knowledge.

The purpose of this paper is to seek 3D fundamental solutions for the contact problem of a half-space punched by a smooth and rigid half-infinite indenter, in the framework of magneto-electro-elasticity. The material is assumed to be transversely isotropic and the indenter may be electrically and magnetically conducting, electrically conducting and magnetically insulating, electrically insulating and magnetically conducting, or both electrically and magnetically insulating. The corresponding boundary value problems are solved by means of the general solutions in conjunction with the method of potential theory, which is generalized to the contact problem within magneto-electro-elasticity *for the first time*. The boundary integral equations for various boundaries, which have the same mathematical structures, are solved by referring to the results available in the literature. The Green's functions of the potentials are exactly derived and the complete fundamental solutions in closed form are explicitly expressed in terms of elementary functions. The singularities of the generalized stresses are examined and the corresponding intensity factors are presented. The analytical solutions in this study can not only serve as benchmarks to simplified analyses and numerical methods, but also play an important role in characterizing the physical properties of multiferroic composites.

2. Basic equations and general solutions

In the Cartesian coordinate system (x, y, z) with the z-axis normal to the isotropic plane, the constitutive relations of transversely isotropic MEE materials read (Ding et al., 1997a; Chen et al., 2010)

$$
\begin{aligned}
\sigma_x &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} + d_{31}\frac{\partial \Psi}{\partial z},\\
\sigma_y &= c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} + d_{31}\frac{\partial \Psi}{\partial z},\\
\sigma_z &= c_{13}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \Phi}{\partial z} + d_{33}\frac{\partial \Psi}{\partial z},\\
\tau_{yz} &= c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + e_{15}\frac{\partial \Phi}{\partial y} + d_{15}\frac{\partial \Psi}{\partial y},\\
\tau_{zx} &= c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \Phi}{\partial x} + d_{15}\frac{\partial \Psi}{\partial x},\\
\tau_{xy} &= c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned}
\tag{1a}
$$

$$
\begin{aligned}
D_x &= e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial x} - g_{11}\frac{\partial \Psi}{\partial x},\\
D_y &= e_{15}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial y} - g_{11}\frac{\partial \Psi}{\partial y},\\
D_z &= e_{31}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z} - g_{33}\frac{\partial \Psi}{\partial z},
\end{aligned}
\tag{1b}
$$

$$
\begin{aligned}
B_x &= d_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - g_{11}\frac{\partial \Phi}{\partial x} - \mu_{11}\frac{\partial \Psi}{\partial x},\\
B_y &= d_{15}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - g_{11}\frac{\partial \Phi}{\partial y} - \mu_{11}\frac{\partial \Psi}{\partial y},\\
B_z &= d_{31}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + d_{33}\frac{\partial w}{\partial z} - g_{33}\frac{\partial \Phi}{\partial z} - \mu_{33}\frac{\partial \Psi}{\partial z},
\end{aligned}
\tag{1c}
$$

where $\sigma_i(\tau_{ij})$, D_i and B_i are stress, electric displacement and magnetic induction components, respectively; $u(v, w)$, Φ and Ψ are the elastic displacements, electric potential and magnetic potential, respectively, which are referred to as generalized displacements; c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} are respectively the elastic, piezoelectric, piezo-magnetic, dielectric, electromagnetic and magnetic constants. Furthermore, we have an additional relation $2c_{66} = c_{11} - c_{12}$ for media with transverse isotropy. It is evident that various decoupled cases can be degenerated from (1) by letting the corresponding coupling constants vanish.

Without the effect of body forces, electric and "magnetic" charges, the generalized equilibrium equations are

$$
\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,\\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,\\
&\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0,
\end{aligned}
\tag{2a}
$$

$$
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0,
\tag{2b}
$$

$$
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.
\tag{2c}
$$

Substituting (1) into (2), we can derive the equilibrium equations in terms of generalized displacements, for which the general solutions were proposed by Ding and Jiang (2003) and Hou et al. (2005) by means of the rigorous operator theory and the generalized Almansi theorem. The form of the general solutions depends on the following algebraic equation

$$
n_0 s^8 - n_1 s^6 + n_2 s^4 - n_3 s^2 + n_4 = 0,
\tag{3}
$$

where the coefficients n_i $(i = 0, 1, \ldots, 4)$ are given in Ding and Jiang (2003) and Hou et al. (2005), and are listed in Appendix. From a mathematical point of view, (3) is the characteristic equation of an elliptical partial differential equation of the 8*th* order, which is satisfied by a potential function. In that partial differential equation, derivatives of odd orders with respect to the variable z do not appear (Ding and Jiang, 2003; Hou et al., 2005). For the piezoelectric, piezo-magnetic and elastic materials as special cases, the corresponding eigen-equation can be reduced from (3) as shown in Chen et al. (2010). In the present study, the eigenvalues s_i in (3) have a real part, whose correlation determines the form of the general solutions in terms of quasi-harmonic functions. In what follows, our concern is confined only to transversely isotropic media with distinct eigenvalues. In this case, the general solutions are of the simplest form.

Introducing the following complex variables with $\mathrm{i} = \sqrt{-1}$

$$
\begin{aligned}
&U = u + \mathrm{i}v, \quad w_1 = w, \quad w_2 = \Phi, \quad w_3 = \Psi,\\
&\sigma_1 = \sigma_x + \sigma_y, \quad \sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy}, \quad \tau_{z1} = \tau_{zx} + \mathrm{i}\tau_{yz}, \quad \sigma_{z1} = \sigma_z,\\
&\tau_{z2} = D_x + \mathrm{i}D_y, \quad \tau_{z3} = B_x + \mathrm{i}B_y, \quad \sigma_{z2} = D_z, \quad \sigma_{z3} = B_z,
\end{aligned}
\tag{4}
$$

Ding and Jiang (2003) obtained the following general solutions in compact form

$$
\begin{aligned}
&U = \Lambda\left(\sum_{j=1}^{4}\psi_j + \mathrm{i}\psi_0\right), \quad w_m = \sum_{j=1}^{4} s_j k_{mj}\frac{\partial \psi_j}{\partial z_j} \quad (m = 1-3)\\
&\sigma_1 = 2\sum_{j=1}^{4}(c_{66} - \omega_{1j}s_j^2)\frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_2 = 2c_{66}\Lambda^2\left(\sum_{j=1}^{4}\psi_j + \mathrm{i}\psi_0\right),\\
&\sigma_{zm} = \sum_{j=1}^{4}\omega_{mj}\frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \tau_{zm} = \Lambda\left(\sum_{j=1}^{4} s_j \omega_{mj}\frac{\partial \psi_j}{\partial z_j} + \mathrm{i}s_0\rho_m\frac{\partial \psi_0}{\partial z_0}\right),
\end{aligned}
\tag{5}
$$