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Novel numerical implementation of asymptotic homogenization method for periodic plate structures

Yuanwu Cai, Liang Xu, Gengdong Cheng *

Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China

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ABSTRACT

The present paper develops and implements finite element formulation for the asymptotic homogenization theory for periodic composite plate and shell structures, earlier developed in Kalamkarov (1987) and Kalamkarov (1992), and thus adopts this analytical method for the analysis of periodic inhomogeneous plates and shells with more complicated periodic microstructures. It provides a benchmark test platform for evaluating various methods such as representative volume approaches to calculate effective properties. Furthermore, the new numerical implementation (Cheng et al., 2013) of asymptotic homogenization method of 2D and 3D materials with periodic microstructure is shown to be directly applicable to predict effective properties of periodic plates without any complicated mathematical derivation. The new numerical implementation is based on the rigorous mathematical foundation of the asymptotic homogenization method, and also simplicity similar to the representative volume method. It can be applied easily using commercial software as a *black box*. Different kinds of elements and modeling techniques available in commercial software can be used to discretize the unit cell. Several numerical examples are given to demonstrate the validity of the proposed methods.

1. Introduction

Plate and shell structures are widely used in various applications, such as aerospace, marine, and other engineering applications. To obtain higher stiffness and lighter weight, these structures are often accompanied by stiffeners, ribs, or other complicated microstructures, such as corrugated plate and lattice truss core sandwich panel, which result in inhomogeneous and heterogeneous material in micro scale. Analysis of this kind of structure can be tedious because of their large scale and complex microstructures.

In practical applications, plate and shell structures such as honeycomb plate are often comprised of periodic unit cells. This kind of plate and shell structures are periodic in-plane, and can be considered as homogeneous macroscopically. Many researchers have studied these kinds of structures, and developed various methods to predict their effective stiffnesses, such as engineering approaches (Gibson and Ashby, 1999; Chen, 2011), asymptotic homogenization method (Kalamkarov, 1987, 1992; Kalamkarov and Kolpakov, 1997; Kalamkarov et al., 2009), variational asymptotic method (Lee and Yu, 2011; Xia et al., 2003) assumed that the average mechanical properties of a representative volume element (RVE) are equal to the average properties of the particular composite laminate.

Various engineering approaches are widely used for plate and shell structures for their simplicity. These approaches usually simplify mechanical behavior of the unit cell, which are only applicable when the unit cell and its components have certain relations. For the periodic honeycomb plate, in-plane elastic modulus of the honeycomb formed from hexagonal cells is given in Gibson and Ashby (1999). Bending stiffnesses are obtained using in-plane elastic moduli based on the Kirchhoff hypothesis. However, Chen (2011) found that the honeycomb bending can not be evaluated by using the equivalent elastic moduli obtained from the in-plane deformation, and proposed a theoretical technique for calculating the honeycomb flexural rigidity (Chen, 2011). Other plate configurations are also studied by many researchers, such as re-entrant honeycomb structure (Grima et al., 2011; Scarpa et al., 2000).

Asymptotic homogenization method is a well-known method in predicting material properties. It has rigorous mathematical foundation, which is based on perturbation theory, and calculates effective properties by solving partial differential equations defined on a unit cell (see Hassani and Hinton (1999) for reference on prediction of effective property of material with periodicity in three dimensions). This theory gives exact solution if the macrostructure is large enough so it is composed of a very large number of unit cells. (Kalamkarov, 1987, 1992) (see also Kalamkarov and Kolpakov (1997) and Kalamkarov et al. (2009)) developed asymptotic homogenization theory for plate and shell through elaborate and complicated analytical derivation, which takes thickness of the plate and shell as a small dimension, and has microstructure

* Corresponding author.
E-mail address: chenggd@dlut.edu.cn (G. Cheng).

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periodic in-plane with small tangential dimensions of a periodicity cell comparable with the small thickness. Based on this theory, they analyzed a variety of composite and structurally inhomogeneous plate and shell structures using certain simplifications in the solution of the corresponding unit cell problems, see Kalamkarov (1987, 1992), Kalamkarov and Kolpakov (1997), Kalamkarov et al. (2006, 2007, 2009), Kalamkarov and Georgiades (2004), Georgiades and Kalamkarov (2004) and Georgiades et al. (2010). However, the finite element method is not given any priority in their work (Pedersen, 1998). It is known that this asymptotic homogenization theory of periodic plate structure is difficult to implement numerically (Lee and Yu, 2011).

The present paper develops and implements finite element formulation for the asymptotic homogenization theory for plate and shell developed by Kalamkarov (1987) and Kalamkarov (1992), thus adopts this method to be capable of solving plate and shell with complicated microstructures. Because of the rigorous mathematical foundation of this method, it provides a benchmark test platform for evaluating various approaches such as RVE method to calculate effective properties.

In numerical implementation of finite element method for the asymptotic homogenization theory, only 2D solid element and 3D solid element are mostly adopted in literature. This seems due to the fact that researchers often should do more work on derivation and coding of finite element formulation of the unit cell problem according to different element types. But if we only use these two kinds of elements to discretize all unit cells with complicated microstructures, there will be normally large numbers of elements in one unit cell, especially if components of different sizes exist, and solving these finite element equations is usually time-consuming. The authors have developed a new numerical implementation of the asymptotic homogenization method to predict effective properties of periodic materials with periodicity in three dimensions (Cheng et al., 2013). It is interesting to discover this new numerical implementation can be extended to plate structure with periodicity in-plane without any complicated mathematical derivation. The new numerical implementation has rigorous mathematical foundation of the asymptotic homogenization method, and also simplicity as the RVE method. It can use commercial software as a *black box*, and use all kinds of elements and modeling techniques available in commercial software to discretize the unit cell, so the model may remain a fairly small scale.

This paper is organized as follows. Firstly, the asymptotic homogenization theory for plate and shell developed by Kalamkarov (1987, 1992) is described briefly in Section 2, and its finite element formulation is given in Section 3. The new numerical implementation of the asymptotic homogenization method to predict effective properties of plate with periodicity in-plane is presented in Section 4. After that, Section 5 gives some examples to verify these methods. At last, conclusions are drawn in Section 6.

2. Asymptotic homogenization of periodic plate structure

Plate and shell structure has a small dimension along its thickness direction in comparison with other two directions, and how to derive the theory of plate and shell from the three dimensional elasticity theory is a long-standing and challenging problem in the last century. Asymptotic homogenization theory of 2D and 3D periodic material has already been developed before (Bakhvalov and Panasenko, 1989; Bensoussan et al., 1978; Sanchez-Palencia and Zaoui, 1987). For plate and shell structure with periodic microstructure in-plane and a finite dimension in thickness direction, how to derive their effective properties from the rigorous mathematical homogenization theory is equally difficult. Kalamkarov (1987, 1992) analytically developed asymptotic homogenization theory of plate and shell based on the rigorous perturbation theory.

Consider a general three dimensional layer with periodic microstructure in-plane using the notations and figures in Kalamkarov (1992), Kalamkarov and Kolpakov (1997) and Kalamkarov et al. (2009), see Fig. 1. The periodic unit cell Ω is shown in Fig. 1(b). $\alpha_1, \alpha_2, \gamma$ are orthogonal curvilinear coordinates, such that the coordinate lines α_1, α_2 coincide with the main curvature lines of the middle surface and coordinate line γ is normal to the middle surface ($\gamma = 0$). Thickness of the layer and the dimensions of the unit cell are assumed to be small as compared with the dimensions of the structure in whole. These small dimensions of the periodicity cell are characterized by a small parameter δ.

The unit cell Ω is defined by the following relations:

$$
\begin{aligned}
&-\frac{\delta h_1}{2} < \alpha_1 < \frac{\delta h_1}{2}, \quad -\frac{\delta h_2}{2} < \alpha_2 < \frac{\delta h_2}{2}, \quad \gamma^- < \gamma < \gamma^+, \\
&\gamma^\pm = \pm\frac{\delta}{2} \pm \delta F^\pm\left(\frac{\alpha_1}{\delta h_1}, \frac{\alpha_2}{\delta h_2}\right)
\end{aligned} \tag{1}
$$

where δ is the thickness of the layer, δh_1 and δh_2 are dimensions of the unit cell in middle surface, functions $F^\pm$ define the geometry of the upper and lower reinforcing elements S^+, S^-, such as ribs and stiffeners.

Introduce the following fast variables $\xi = (\xi_1, \xi_2), z$:

$$
\xi_1 = \frac{\alpha_1 A_1}{\delta h_1}, \quad \xi_2 = \frac{\alpha_2 A_2}{\delta h_2}, \quad z = \frac{\gamma}{\delta} \tag{2}
$$

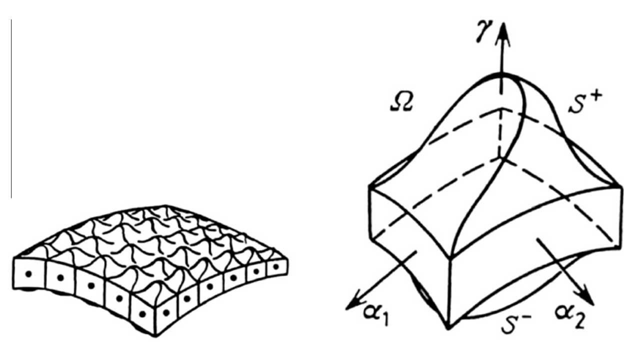
where $A_1(\boldsymbol{\alpha}), A_2(\boldsymbol{\alpha})$ are *Lamé coefficients* at point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ on the middle surface in α_1, α_2 directions respectively.

The displacements and stresses are expressed in the form of the following two-scale asymptotic expansions:

$$
\begin{aligned}
u_i(\boldsymbol{\alpha}, \xi, z) &= u_i^{(0)}(\boldsymbol{\alpha}) + \delta u_i^{(1)}(\boldsymbol{\alpha}, \xi, z) + \delta^2 u_i^{(2)}(\boldsymbol{\alpha}, \xi, z) + \cdots \\
\sigma_{ij}(\boldsymbol{\alpha}, \xi, z) &= \sigma_{ij}^{(0)}(\boldsymbol{\alpha}, \xi, z) + \delta\sigma_{ij}^{(1)}(\boldsymbol{\alpha}, \xi, z) + \delta^2\sigma_{ij}^{(2)}(\boldsymbol{\alpha}, \xi, z) + \cdots
\end{aligned} \tag{3}
$$

Here Latin indices i, j assume values of 1, 2, 3. Substitute (3) into the equilibrium equations, and expand in powers of δ. As a result of asymptotic homogenization procedure, the following relations between displacements and stresses can be drawn (Kalamkarov, 1992; Kalamkarov and Kolpakov, 1997; Kalamkarov et al., 2009):

$$
\begin{aligned}
u_1 &= v_1(\boldsymbol{\alpha}) - \frac{\gamma}{A_1}\frac{\partial w(\boldsymbol{\alpha})}{\partial \alpha_1} + \delta U_1^{\mu\nu}\varepsilon_{\mu\nu} + \delta^2 V_1^{\mu\nu}\tau_{\mu\nu} + O(\delta^3) \\
u_2 &= v_2(\boldsymbol{\alpha}) - \frac{\gamma}{A_2}\frac{\partial w(\boldsymbol{\alpha})}{\partial \alpha_2} + \delta U_2^{\mu\nu}\varepsilon_{\mu\nu} + \delta^2 V_2^{\mu\nu}\tau_{\mu\nu} + O(\delta^3) \\
u_3 &= w(\boldsymbol{\alpha}) + \delta U_3^{\mu\nu}\varepsilon_{\mu\nu} + \delta^2 V_3^{\mu\nu}\tau_{\mu\nu} + O(\delta^3) \\
\sigma_{ij} &= b_{ij}^{\mu\nu}\varepsilon_{\mu\nu} + \delta b_{ij}^{*\mu\nu}\tau_{\mu\nu} + O(\delta^2)
\end{aligned} \tag{4}
$$



(a) (b)

Fig. 1. (a) Three dimensional composite layer with periodic microstructure in-plane. (b) Unit cell. (Kalamkarov, 1992; Kalamkarov and Kolpakov, 1997; Kalamkarov et al., 2009)