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Nonlocal elasticity defined by Eringen's integral model: Introduction of a boundary layer method

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ABSTRACT

In this paper we consider a nonlocal elasticity theory defined by Eringen's integral model and introduce, for the first time, a boundary layer method by presenting the exponential basis functions (EBFs) for such a class of problems. The EBFs, playing the role of the fundamental solutions, are found so that they satisfy the governing equations on an unbounded domain. Some insight to the theory is given by showing that the EBFs satisfying the Navier equations in the classical elasticity theory also satisfy the governing equations in the nonlocal theory. Some additional EBFs are particularly obtained for the nonlocal theory. In order to use the EBFs on bounded domains, the effects of the boundary conditions are taken into account by truncating the kernel/attenuation function in the constitutive equations. This leads to some residuals in the governing equations which appear near the boundaries. A weighted residual approach is employed to minimize the residuals near the boundaries. The method presented in this paper has much in common with Trefftz methods especially when the influence area of the kernel function is much smaller than the main computational domain. Several one/two dimensional problems are solved to demonstrate the way in which the EBFs can be used through the proposed boundary layer method.

1. Introduction

Nonlocal elasticity models have received considerable attention by the researches intending to design or analyze Micro/Nano structures. The models extend the main concepts in the classical theory of elasticity to approximate the behavior of particles, as small as molecules or atoms, and therefore they play the role of models with bridging scales in the analysis of multi-scale problems. According to Eringen (1987), lack of an internal characteristic length in the classical theory limits the application of this theory in the modeling of physical problems in which the influence of microstructural effects is significant. The first attempts to modify the continuum approaches were made by Kröner (1967), Kunin (1984) and Krumhansl (1968). Improved formulations were proposed later by Edelen and Laws (1971), Edelen et al. (1971) and Eringen and Edelen (1972). Extensive studies by Eringen and Kim (1974) and Eringen et al. (1977) on nonlocal elasticity problems, with linear homogeneous and isotropic materials, must be mentioned here. The readers can find comprehensive surveys of nonlocal plasticity and damage models in the review papers by Bažant and Jirásek (2002) or Jirásek and Rolshoven (2003). More investigations on the choice of kernel function in nonlocal damage problems can be found in the studies by Borino et al. (2003).

Similar to the cases in the classical theories, the exact solution of problems defined with a nonlocal theory is almost impossible to achieve except for very few 1D cases. The readers may refer to the early studies by Pisano and Fuschi (2003), and more accurate and complete ones by Challamel and Wang (2008), Challamel et al. (2009a,b) and Benvenuti and Simone (2013). With the lack of analytical solutions, the use of numerical methods seems to be inevitable. The use of finite element method (FEM) has been reported in the studies by Polizzotto (2001) and Pisano et al. (2009). The studies by Schwartz et al. (2012) on the application of the boundary element method (BEM) should be mentioned here.

As indicated in almost all the aforementioned studies, the numerical solution of nonlocal elasticity problems is very time consuming. Therefore any mesh reduction approach may be considered vital in order to reduce the computational time. Nevertheless, as the numerical simulation tools are advancing, the lack of benchmark problems to access the capabilities of the numerical methods is increasingly felt especially in multi-dimensional cases. The objective of the recent studies by the authors is to attain such a goal (see Abdollahi and Boroomand, 2013). The paper presents a series of low-residual solutions for 1D and 2D problems using Chebyshev polynomials. Such polynomials proved to be useful in the solution of many problems in physics and engineering (see

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Boyd, 2000) but of course may not play the role of fundamental solutions in nonlocal elasticity problems. Therefore, in order to solve the benchmark problems, some energy principles were used which require writing integral equations over the solution domain leading to a very expensive procedure (considering the integrals needed in the constitutive relations). Since the approach followed in that reference is extremely time consuming, the results of the benchmark solutions were given explicitly through a series of tables. The same approach may be followed for problems in 3D but of course the computational cost will be enormous.

In seeking low-residual solutions, one may think of using a Trefftz approach, as in the method of fundamental solutions (MFS) (see Kupradze and Aleksidze, 1964 or Fairweather and Karageorghis, 1998) or the BEM (see Brebbia and Dominguez, 1998) for instance, in which a series of predefined fundamental basis functions satisfying the governing equations are used (see also Zielinski and Herrera, 1987, Kita, 1995, Chen et al., 2009 for more details of Trefftz methods). However, the main hurdle in this way is the evaluation of the bases or the Green's functions. This paper deals with such a concept. After finding the fundamental bases, they may be used in a variety of simulation tools (letting alone the insight provided by them) including the one presented in this paper.

In this paper we first introduce some fundamental basis functions which fully satisfy the governing equations in a nonlocal elasticity theory defined by an integral constitutive low on unbounded domains. The bases are found in the form of exponential functions with complex exponents. The readers may find the application of similar bases to other engineering problems in the studies by Boroomand et al. (2010), Shamsaei and Boroomand (2011), Shahbazi et al. (2011a,b, 2012) and Azhari et al. (2013a,b) for static and time harmonic problems and the works by Zandi et al. (2012a,b), Hashemi et al. (2013), Movahedian et al. (2013) and Movahedian and Boroomand (2014) for transient problems (see also the extension of the method in Boroomand and Noormohammadi, 2013 for more general elasticity problems). In deriving such exponential basis functions (EBFs) we show that some of the bases are identical to those found for the classical elasticity theory earlier by the second author and co-workers (Boroomand et al., 2010). However, there are also some additional bases which are particularly found for the nonlocal theory. We present the closed form of the EBFs for 1D/2D problems using various attenuation/kernel functions. This gives an insight to the behavior of the material with nonlocal constitutive laws.

Having found the EBFs, we proceed to use them in a boundary layer approach. The term "boundary layer" stems from the fact that the use of the EBFs, defined on unbounded domains, in a bounded problem produces some residuals in a region close to the boundaries. To reduce such residuals, we employ a numerical approach with weights defined just on the boundary layer zone. With such features, the method may be classified in the Trefftz type of methods, similar to the BEM or MFS.

Letting alone the achievable accuracy by Trefftz methods, as is the case for the boundary layer method proposed here, the computational time saved by using them may be expected to be similar to the save of time when BEM and FEM codes are used. This especially becomes noticeable in the solution of problems defined on large domains with relatively small influence distance of nonlocal effect. The importance of this effect may be best understood when 3D problems are of concern. This, however, is beyond the scope of this paper.

In this paper, we demonstrate the capabilities of the method, in some 1D/2D problems, by comparing the results with the benchmark problems recently provided by the authors (Abdollahi and Boroomand, 2013) using Chebyshev polynomials.

The layout of the paper is as follows. In the next section we present an overview of the theory used in this paper. The way that the EBFs are extracted from the governing equation is explained in Section 3. In Section 4 we discuss on the choice of the attenuation/kernel function used. Section 5 is devoted to the introduction of a boundary layer method using the EBFs found. The numerical experiments are presented in Section 6 where we validate the results for 1D/2D problems. The conclusions made throughout the paper are summarized in Section 7.

2. Nonlocal model; an overview

We consider an elastic body occupying Ω in a 1D/2D space. The equilibrium equations in the local/nonlocal elasticity problems are written as

$$\mathbf{S}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega. \tag{1}$$

The following boundary conditions are also considered

$$\mathbf{u} = \mathbf{u}_B \quad \text{on } \Gamma_u, \tag{2}$$

and

$$\tilde{\mathbf{n}}\boldsymbol{\sigma} = \mathbf{t} \quad \text{on } \Gamma_t. \tag{3}$$

In the above relations $\boldsymbol{\sigma}$ is the vector of stresses, $\mathbf{u}$ is the vector of displacements, $\mathbf{S}$ is the well-known operator for defining the strains as $\boldsymbol{\varepsilon} = \mathbf{S}\mathbf{u}$, $\mathbf{b}$ is the vector of body force, $\mathbf{u}_B$ and $\mathbf{t}$ are the boundary displacement and tractions, respectively, and $\tilde{\mathbf{n}}$ is a matrix containing the components of the unit vector normal to the boundary for defining the tractions.

According to Eringen's model (2002) the stresses at a generic point as $\mathbf{x} = [x, y]^T$ are dependent on the strains at other points of the domain, here known as $\mathbf{x}' = [x', y']^T$. The strain and stress fields satisfy the following constitutive integral equation

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_\Omega k(\mathbf{x}', \mathbf{x})\, \mathbf{D}\, \boldsymbol{\varepsilon}(\mathbf{x}')\, \mathrm{d}\Omega_{\mathbf{x}'} \quad \forall \mathbf{x}, \mathbf{x}' \in \Omega. \tag{4}$$

In the above relation $\mathbf{D}$ is the matrix of material constants as in the classical elasticity theory which is generally written as

$$\mathbf{D} = \begin{pmatrix} D_1 & D_2 & 0 \\ D_2 & D_1 & 0 \\ 0 & 0 & D_3 \end{pmatrix} \tag{5}$$

for 2D problems. The attenuation/kernel function $k(\mathbf{x}', \mathbf{x})$ plays the role of a measure for the dependence of the stresses at $\mathbf{x}$ to the strains at $\mathbf{x}'$ (in (4) $\mathrm{d}\Omega_{\mathbf{x}'}$ denotes the volume fraction at $\mathbf{x}'$). When isotropy is of concern, which is the case in this study, $k(\mathbf{x}', \mathbf{x})$ is written as a function of the distance between $\mathbf{x}$ and $\mathbf{x}'$, i.e.

$$k(\mathbf{x}', \mathbf{x}) = k(|\mathbf{x}' - \mathbf{x}|). \tag{6}$$

We use such a form in the rest of the formulation given in this paper. The function is chosen so that it reaches to its maximum at $\mathbf{x} = \mathbf{x}'$ and attenuates for large distances between $\mathbf{x}$ and $\mathbf{x}'$, i.e.

$$\lim_{|\mathbf{x}' - \mathbf{x}| \to \infty} k(|\mathbf{x}' - \mathbf{x}|) = 0, \tag{7}$$

and also

$$\int_{\Omega_\infty} k(|\mathbf{x}' - \mathbf{x}|)\, \mathrm{d}\Omega = 1, \tag{8}$$

analogous to a Dirac delta function, e.g. when a very sharp kernel function is used to recover the constitutive relations in the classical theory (in (8) Ω_∞ denotes an unbounded domain). It is clear that the sharpness of $k(|\mathbf{x} - \mathbf{x}'|)$ represents an internal characteristic length for the material.