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Stress gradient versus strain gradient constitutive models within elasticity

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ABSTRACT

A stress gradient elasticity theory is developed which is based on the Eringen method to address nonlocal elasticity by means of differential equations. By suitable thermodynamics arguments (involving the free enthalpy instead of the free internal energy), the restrictions on the related constitutive equations are determined, which include the well-known Eringen stress gradient constitutive equations, as well as the associated (so far uncertain) boundary conditions. The proposed theory exhibits complementary characters with respect to the analogous strain gradient elasticity theory. The associated boundary-value problem is shown to admit a unique solution characterized by a Hellinger–Reissner type variational principle. The main differences between the Eringen stress gradient model and the concomitant Aifantis strain gradient model are pointed out. A rigorous formulation of the stress gradient Euler–Bernoulli beam is provided; the response of this beam model is discussed as for its sensitivity to the stress gradient effects and compared with the analogous strain gradient beam model.

1. Introduction

Eringen (1983) proposed a method to address boundary-value problems of nonlocal (integral) elasticity whereby the inherent integro-differential equations are replaced by differential equations. The method is grounded on the constitutive relation

$$\underbrace{\mathbf{C}:\boldsymbol{\varepsilon}}_{\mathbf{s}} = \underbrace{\boldsymbol{\sigma} - \ell^2\Delta\boldsymbol{\sigma}}_{\mathrm{L}\boldsymbol{\sigma}} \tag{1}$$

where ℓ is a material constant with the meaning of internal length scale parameter, Δ is the Laplacian operator and $\mathbf{C}$ is the usual fourth order moduli tensor of isotropic elasticity. With the language of nonlocal elasticity, (1) can be qualified as a differential relation (featured by the operator $L := 1 - \ell^2\Delta$) between the *nonlocal* stress field $\boldsymbol{\sigma}$ and the *local* strain field $\boldsymbol{\varepsilon}$ (or the associated *local* Hookean stress $\mathbf{s} := \mathbf{C} : \boldsymbol{\varepsilon}$), that is, between two fields which more naturally are related through an integral-type relation as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|)\mathbf{s}(\mathbf{x}')\,\mathrm{d}v(\mathbf{x}') \tag{2}$$

Here, V is the material domain and $\alpha(|\mathbf{x}' - \mathbf{x}|)$ is the *influence function* (Eringen, 2002). This is a positive function of the distance $|\mathbf{x}' - \mathbf{x}|$ between the field point $\mathbf{x}$ and the source point $\mathbf{x}'$; it has a maximum value at $\mathbf{x}' = \mathbf{x}$ and decays more or less rapidly with the increasing distance $|\mathbf{x}' - \mathbf{x}|$, becoming vanishing at all points $\mathbf{x}'$ located out of a sphere of (relatively small) radius R and centered at $\mathbf{x}$. The equivalence between (1) and (2) stems from the restriction that α is the Green function of the operator L. In fact, on applying the latter operator to (2), since $\mathrm{L}\alpha = \delta_{\mathrm{D}} =$ Dirac delta, (1) can be readily obtained.

The nonlocal stress field $\boldsymbol{\sigma}$ is required to satisfy the standard equilibrium equations, namely,

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V, \quad \mathbf{n}\cdot\boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } S_f \tag{3}$$

where $\mathbf{b}$ denotes body forces within V and $\bar{\mathbf{t}}$ surface forces assigned over the free part S_f of the boundary surface $S = \partial V$; $\mathbf{n}$ is the unit outward normal to S. The latter body forces are presumed to include the inertia forces, if any. Furthermore, the local strain field $\boldsymbol{\varepsilon}$ is required to satisfy the standard compatibility equations, that is,

$$\boldsymbol{\varepsilon} = \nabla^s\mathbf{u} \quad \text{in } V, \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } S_c \tag{4}$$

where ∇^s denotes the symmetric part of the gradient operator ∇, whereas $\bar{\mathbf{u}}$ is the imposed displacement on the constrained part $S_c = S \setminus S_f$ of S.

The Eringen method consists in associating (3) and (4) with the differential constitutive equations of (1) instead of the integral type ones of (2). This means that the *non-locality effects* of the original integral-type problem enter into play within the

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differential-type problem as *gradient effects* originating from a source identified with the Cauchy stress $\boldsymbol{\sigma}$. In other words, the original nonlocal integral-type model is replaced with a *stress gradient model*. Indeed the latter model finds itself in strong contrast with the well-known *strain gradient model* widely employed to describe size effects and other phenomena of small scale solids.

The popularity of the above Eringen method stems from the relative easiness with which a differential type boundary-value problem can be solved with respect to one of integro-differential nature. Indeed, on combining (1), (3) and (4), one easily obtains the following displacement equation

$$\mathcal{L}\mathbf{u} = -\mathbf{b}^* \quad \text{in } V \tag{5}$$

where

$$\mathbf{b}^* := \mathrm{L}\mathbf{b} = \mathbf{b} - \ell^2 \Delta \mathbf{b} \tag{6}$$

The symbol $\mathcal{L}$ denotes the classical set of second-order partial differential equations (PDEs) of isotropic elasticity, that is, denoting by λ and μ the Lamé constants,

$$\mathcal{L}\mathbf{u} := \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{u}. \tag{7}$$

The PDE system (5) has to be solved in association with the boundary conditions $(3)_2$ and $(4)_2$, of which the former carries in the stress $\boldsymbol{\sigma}$. This implies that the obtained boundary-value problem cannot in general be solved without considering the PDEs (1) together with the associated boundary conditions (heuristically devised, since their exact form is unknown from the wide literature, to the author's knowledge).

The above method has been widely used to address problems within nanotechnology, crack problems at the microscale, dislocation analysis within unbounded domains, etc. For more complex problems an approximation can be taken by replacing the coupling boundary condition $(3)_2$ with a similar uncoupling one in which the nonlocal stress $\boldsymbol{\sigma}$ is replaced by the local stress $\mathbf{s}$. In this way the resulting boundary-value problem identifies—except for the body force, if any—with the classical one, whereas the related stress field $\boldsymbol{\sigma}$ may then be obtained by solving the PDEs (1), (indeed, an operation which requires due care since there cannot be any guarantee that the derived stress field satisfies the equilibrium equations). It is not the purpose of the present paper to review the extensive literature on this topic; we just mention some representative works and the references therein, namely Eringen (1983, 2002), Lazar et al. (2006a,b), Askes and Gutiérrez (2006), Reddy (2007), Reddy and Pang (2008), Peddieson et al. (2003), Kumar et al. (2008).

The present paper is more interested in other aspects of the Eringen method, emerging when the stress gradient model discussed previously is compared with a strain gradient model featured by a constitutive equation similar to (1), that is,

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \ell^2 \Delta\boldsymbol{\varepsilon}) \tag{8}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress satisfying the equilibrium equations (3) and $\boldsymbol{\varepsilon}$ is the standard strain satisfying the compatibility equations (4). The gradient elasticity model based on (8)—often referred to as the Aifantis elasticity model (Aifantis, 1992; Ru and Aifantis, 1993; Altan and Aifantis, 1997)—can be viewed as a particularization of a more general strain gradient model devised by Mindlin (1965), Mindlin and Eshel (1968), Wu (1992); see Askes and Aifantis (2011) for an overview on the latter models. On comparing (8) and (1) with each other, one can observe that the stress gradient model (1) exhibits a character of *complementarity* (in the mechanical sense) with respect to the strain gradient model (8). However, whereas the thermodynamic consistency of (8) as a gradient constitutive model has been already assessed within the literature (see e.g., Polizzotto, 2011 and the literature therein), no such investigations seem to exist for (1). It is therefore quite natural to raise the following question:

Is there any thermodynamics-based procedure which, like the analogous procedures devised for the strain gradient models, may lead to the Eringen constitutive equation (1) and to the related boundary conditions?

The main purpose of the present paper is to give a positive answer to the latter question. Indeed, it will be shown that any thermodynamics-based procedure devised for a strain gradient model, but suitably changed into one of complementary nature, may constitute a procedure suitable to cope with a stress gradient model. This requires that (i) the principle of the virtual power (PVP) (for velocities) must be replaced with the complementary PVP (for stress rates), and (ii) the internal energy and the (Helmholtz) free energy must be replaced with the enthalpy and the (Gibbs) free enthalpy, respectively.

The outline of the paper is as follows. In Section 2, some thermodynamics premises are developed in the purpose to obtain a complementary form of the Clausius–Duhem inequality expressed in terms of the Gibbs function, that is, a thermodynamic potential depending on the stress, the temperature and, possibly, the stress gradient. In Section 3, an extended form of the principle of the virtual power (PVP) for stress gradient materials is presented, which is the complementary counterpart of the analogous PVP for strain gradient materials well-known from the literature (Mindlin, 1965; Germain, 1973), and which leads to the higher order compatibility equations. In Section 4, the results derived in the preceding sections are used to determine the restrictions on the constitutive equations for a stress gradient material; as an alternative to the PVP, a complementary form of the so-called energy residual can be used. The obtained restrictions include the constitutive equations (coinciding with the Eringen stress differential equations (1)), as well as the related boundary conditions in the form $\partial_n \sigma_{ij} = 0$ at all points of the boundary surface. In Section 5 the boundary-value problem associated to the Eringen stress gradient model is addressed and shown to admit a unique solution characterized by two variational principles. One of the latter principles is a minimum principle for the problem to evaluate the stress field associated to a specified strain field through the gradient stress–strain relation and related boundary conditions; the other is a Hellinger–Reissner type principle for the stress and displacement response of a structure subjected to given loads. For comparison purposes, the boundary-value problem associated to a particular class of strain gradient materials (Aifantis elasticity model) is addressed in Section 6, pointing out the main differences between the two models. In Section 7 the Hellinger–Reissner principle is used to derive a complete theory for the Euler–Bernoulli beam, which is discussed in contrast to the analogous strain gradient beam model. Conclusions are drawn in Section 8.

Notation. A compact notation is used, with boldface letters denoting vectors or tensors of any order. The scalar product between vectors or tensors is denoted with as many dots as the number of contracted index pairs. For instance, denoting by $\mathbf{u} = \{u_i\}, \mathbf{v} = \{v_i\}$, $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}, \boldsymbol{\sigma} = \{\sigma_{ij}\}, \boldsymbol{\tau} = \{\tau_{ijk}\}$ and $\mathbf{A} = \{A_{ijkh}\}$ some vectors and tensors, one can write: $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, $\mathbf{A} : \boldsymbol{\varepsilon} = \{A_{ijkh}\varepsilon_{kh}\}$, $\mathbf{A} \vdots \boldsymbol{\tau} = \{A_{ijkh}\tau_{jkh}\}$, $\mathbf{A}^T \vdots \boldsymbol{\tau} = \{A_{ijkh}\tau_{kji}\}$. The summation rule for repeated indexes holds and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system, say $\mathbf{x} = (x_1, x_2, x_3)$. The tensor product is simply indicated as, for instance, $\mathbf{uv} = \{u_i v_j\}$, and thus $\mathbf{A} : \mathbf{uv} = \{A_{ijkh} u_k v_h\}$. An upper dot over a symbol denotes its (material) time derivative, $\dot{\mathbf{u}} = \mathrm{d}\mathbf{u}/\mathrm{d}t$. The symbol ∇ denotes the spatial gradient operator, i.e., $\nabla\mathbf{u} = \{\partial_i u_j\}$, ∇^s is the symmetric part of ∇, and Δ is the Laplacian operator. The symbol $:=$ means equality by definition. Other symbols will be defined in the text at their first appearance.