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An improved exponential transformation for nearly singular boundary element integrals in elasticity problems

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ABSTRACT

This paper presents an improved exponential transformation for nearly singular boundary element integrals in elasticity problems. The new transformation is less sensitive to the position of the projection point compared with the original transformation. In our work, the conventional distance function is modified into a new form in the polar coordinate system. Based on the refined distance function, an improved exponential transformation is proposed in the polar coordinate system. Moreover, to perform integrations on irregular elements, an adaptive integration scheme considering both the element shape and the projection point associated with the improved transformation is proposed. Furthermore, when the projection point is located outside the integration element, another nearest point is introduced to subdivide the integration elements into triangular or quadrilateral patches of fine shapes. Numerical examples are presented to verify the proposed method. Results demonstrate the accuracy and efficiency of our method.

1. Introduction

Dealing with singular integrals and nearly singular integrals has been a seemingly daunting task since the early days of the boundary element method (BEM) (Aliabadi et al., 1985; Aliabadi, 2002; Cheng and Cheng, 2005; Cruse and Aithal, 1991, 1993; Liu and Rudolphi, 1999). In this study, we focus on the nearly singular integrals.

Near singularities are involved in many BEM analyses of engineering problems, such as problems on thin shell-like structures (Krishnasamy et al., 1994; Liu, 1998), the crack problems (Dirgantara and Aliabadi, 2000; Sladek et al., 1993a,b), the contact problems (Aliabadi and Martin, 2000), as well as the sensitivity problems (Zhang et al., 1999). Accurate and efficient evaluation of nearly singular integrals with various kernel functions of the type $O(1/r^{\chi})$ is crucial for the successful implementation of the boundary type numerical methods based on boundary integral equations (BIEs), such as the boundary element method (BEM), the boundary face method (BFM) (Zhang et al., 2009a). A near singularity arises when a source point is close to but not on the integration elements. Although those integrals are actually regular in nature, they cannot be evaluated accurately by the standard Gaussian quadrature. This is because, the denominator r, the distance between the source and the field point, is close to zero but not zero. The difficulty encountered in the numerical evaluation mainly results from the fact that the integrands of nearly singular integrals vary drastically with respect to the distance r. Various numerical techniques have been developed to remove the near singularities, such as Taylor expansion algorithm (Mi and Aliabadi, 1996), global regularization (Sladek et al., 1993a,b; Liu and Rudolphi, 1999), coordinate optimization transformation (Sladek et al., 2000), semi-analytical or analytical integral formulas (Niu and Zhou, 2004; Niu et al., 2005; Zhou et al., 2007, 2008), the sinh transformation (Johnston and Elliott, 2005; Johnston et al., 2007; Gu et al., 2013), polynomial transformation (Tells, 1987), adaptive subdivision method (Gao and Davies, 2000; Zhang et al., 2009a), distance transformation technique (Ma and Kamiya, 2001, 2002; Qin et al., 2011), the PART method (Hayami and Matsumoto, 1994; Hayami, 2005), and the exponential transformation (Xie et al., 2011; Zhang et al., 2009b, 2010). Most of them benefit from the strategies for computing singular integrals (Sladek and Sladek, 1992; Sladek et al., 2001). Among those techniques, the exponential transformation technique seems to be a more promising method for nearly singular integrals. However, the transformation is only limited to 2D boundary element and the accuracy is sensitive to the position of the projection point. In this paper, we develop the exponential transformation technique for the nearly singular integrals in 3D boundary element method. Moreover, our method is less sensitive to the position of the projection point.

In our method, firstly the conventional distance function is reviewed. Then the conventional distance function is modified into a new form. Based on the modified distance function, the

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exponential transformation (Xie et al., 2011; Zhang et al., 2009b, 2010) can be developed to 3D BEM in a new form. Moreover, to perform integrations on irregular elements, the element subdivision technique considering both the shape of integration element and positions of the project point is employed in combination with the improved transformation. Although the element subdivision technique is used, the computational cost is reduced dramatically compared with the conventional element subdivision techniques (Gao and Davies, 2000; Zhang et al., 2009a). Furthermore, in order to get subtriangles or subquadrangles of fine shapes, another nearest point is introduced instead of the projection point when the projection point is located outside the integration element. With our method, the boundary nearly singular integrals of regular or irregular elements can be accurately and effectively calculated. Results demonstrate the accuracy and efficiency of our method. Moreover, our method is less sensitive to the projection of the project point than the conventional exponential transformation method.

This paper is organized as follows. The general form of nearly singular integrals is described in Section 2. Section 3 briefly reviews the distance function in the polar coordinate system and then the distance function is constructed in the polar coordinate system in a new form. In Section 4, the transformations for nearly singular integrals are presented and the element subdivision technique is introduced. Numerical examples are given in Section 5. The paper ends with conclusions in Section 6.

2. General descriptions

In this section, we will give a general form of the nearly singular integrals over 3D boundary elements. First we consider the boundary integral equations for 3D elasticity problems. The well-known self-regular BIE for elasticity problems in 3-D is

$$0 = \int_{\Gamma} (u_j(\mathbf{s}) - u_j(\mathbf{y}))T_{ij}(\mathbf{s},\mathbf{y})d\Gamma - \int_{\Gamma} t_j(\mathbf{s})U_{ij}(\mathbf{s},\mathbf{y})d\Gamma \tag{1}$$

where $\mathbf{s}$ and $\mathbf{y}$ represent the field point and the source point in the BEM, with components $\mathbf{s}_i$ and $\mathbf{y}_i$, $i = 1, 2, 3$, respectively and

$$U_{ij} = \frac{1}{16\pi G(1-v)}[(3-4v)\delta_{ij} + r_{,i}r_{,j}]$$

$$T_{ij} = -\frac{1}{8\pi(1-v)}\left\{\left[\frac{\partial r}{\partial n}(1-2v)\delta_{ij} + r_{,i}r_{,j}\right] - (1-2v)(r_{,i}n_j - r_{,j}n_i)\right\}$$

Eq. (1) is discretized on the boundary Γ by boundary elements $\Gamma_e (e = 1 \sim N)$ defined by interpolation functions. The integral kernels of Eq. (1) become nearly singular when the distance between the source point and integration element is very small compared to the size of integration element. And the integrals in Eq. (1) become nearly singular with different orders, namely, $U_{ij}(\mathbf{s},\mathbf{y})$ with near weak singularity, and $T_{ij}(\mathbf{s},\mathbf{y})$ with near strong singularity. In this paper, we develop the exponential transformation method for various boundary integrals with near singularities of different orders. The new method is detailed in following sections. For the sake of clarity and brevity, we take the following integral as a general form to discuss:

$$I = \int_S \frac{f(\mathbf{x},\mathbf{y})}{r^l} dS, \quad l = 1,2,3 \quad r = \|\mathbf{x}-\mathbf{y}\|_2 \tag{2}$$

where f is a smooth function, $\mathbf{x}$ and $\mathbf{y}$ represent the field point and the source point in BEM, with components $\mathbf{x}_i$ and $\mathbf{y}_i$, respectively. S represents the boundary element. We assume that the source point is close to S, but not on it.

3. Construction of new distance function

3.1. Conventional distance function in polar coordinate system

In this section, we will briefly review the distance function (Ma and Kamiya, 2001, 2002; Qin et al., 2011).

As shown in Fig. 1, employing the first-order Taylor expansion in the neighborhood of the projection point, we have:

$$\begin{aligned} x_k - y_k &= x_k - x_k^c + x_k^c - y_k \\ &= \left.\frac{\partial x_k}{\partial t_1}\right|_{\substack{t_1=c_1\\t_2=c_2}}(t_1-c_1) + \left.\frac{\partial x_k}{\partial t_2}\right|_{\substack{t_1=c_1\\t_2=c_2}}(t_2-c_2) + r_0 n_k(c_1,c_2) + O(\rho^2) \\ &= \rho A_k(\theta) + r_0 n_k(c_1,c_2) + O(\rho^2) \end{aligned} \tag{3}$$

where (c_1, c_2) are the coordinates of the projection point in the local system, (t_1, t_2), $\rho = \sqrt{(t_1-c_1)^2 + (t_2-c_2)^2}$, and $r_0 = \|\mathbf{x}^c - \mathbf{y}\|$ which is the minimum distance from the source point to the element in most cases. n_k represents the component of the unit outward direction to the surface boundary and

$$A_k(\theta) = \left.\frac{\partial x_k}{\partial t_1}\right|_{\substack{t_1=c_1\\t_2=c_2}} \cos\theta + \left.\frac{\partial x_k}{\partial t_2}\right|_{\substack{t_1=c_1\\t_2=c_2}} \sin\theta \tag{4}$$

The distance function is expressed as follows:

$$r^2 = (x_k - y_k)(x_k - y_k) = A_k^2(\theta)\rho^2 + r_0^2 + O(\rho^3) \tag{5a}$$

$$r = \sqrt{A_k^2(\theta)\rho^2 + r_0^2 + O(\rho^3)} \tag{5b}$$

Using Eqs. (5a) and (5b), Eq. (2) can be written as:

$$I = \int_{\Gamma} \frac{f(\mathbf{x},\mathbf{y})}{r^l} d\Gamma = \sum_m \int_{\theta_m}^{\theta_{m+1}} \int_0^{\rho(\theta)} \frac{g(\rho,\theta)}{(\rho^2 + \omega^2(\theta))^{l/2}} d\rho d\theta \tag{6}$$

where $\omega(\theta) = \frac{r_0}{A(\theta)}$, $A(\theta) = \sqrt{A_k(\theta)A_k(\theta)}$, and $g(\rho,\theta)$ is a smooth function.

3.2. Improved distance function in polar coordinate

The conventional distance function has been reviewed in Section 3.1. However, as illustrated in Fig. 2, if the projection point is not the ideal point, the line with end points $\mathbf{x}_c$ and $\mathbf{y}$ is not perpendicular to the tangential plane through $\mathbf{x}_c$.

Using Eqs. (3) and (5a), the real distance between the source point and the field points can be written as:

$$r^2 = (x_k - y_k)(x_k - y_k) = a\left[\left(\rho + \frac{b}{2a}\right)^2 + \frac{r_0^2}{a} - \left(\frac{b}{2a}\right)^2 + O(\rho^3)\right] \tag{7}$$

where $a = A_k^2(\theta) > 0$, $b = 2d_k A_k(\theta)$, $r_0^2 = |\mathbf{d}|^2$

The following distance function can be given as:

$$r = \sqrt{a}\sqrt{\left[\left(\rho + \frac{b}{2a}\right)^2 + \delta^2 + O(\rho^3)\right]} \tag{8}$$

where $\delta^2 = \frac{r_0^2}{a} - \left(\frac{b}{2a}\right)^2$

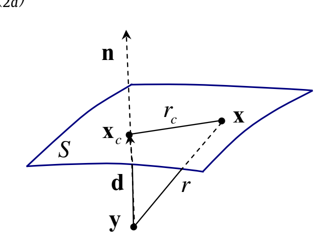

Fig. 1. The minimum distance r_0, from the source point to the projection point x_c the 3D surface element.