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The torsion of a composite, nonlinear-elastic cylinder with an inclusion having initial large strains

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ABSTRACT

This article considers a static problem of torsion of a cylinder composed of incompressible, nonlinear-elastic materials at large deformations. The cylinder contains a central, round, cylindrical inclusion that was initially twisted and stretched (or compressed) along the axis and fastened to a strainless, external, hollow cylinder. The problem statement and solution are based on the theory of superimposed large strains. An accurate analytical solution of this problem based on the universal solution for the incompressible material is obtained for arbitrary nonlinear-elastic isotropic incompressible materials. The detailed investigation of the obtained solution is performed for the case in which the cylinders are composed of Mooney-type materials. The Poynting effect is considered, and it is revealed that composite cylinder torsion can involve both its stretching along the axis and compression in this direction without axial force, depending on the initial deformation.

1. Introduction

When solving problems of the nonlinear theory of elasticity at large strains in the case in which a body is loaded at several steps, it is reasonable to use a mathematical tool of the theory of repeatedly superimposed large strains (Levin, 1998; Levin and Zingerman, 1998, 2008; Zingerman and Levin, 2009) when the problem is stated and solved. For example, in Levin (1998) and Levin and Zingerman (1998, 2008) this theory is used to solve the problems of the stressed-deformed state of the previously loaded nonlinear-elastic bodies with original holes in them. In Zingerman and Levin (2009), an approximate solution to the problems of the origin of an inclusion (a region with the other properties) in a loaded body is obtained. In the present article, a static problem of torsion of the cylinder composed of incompressible, nonlinear-elastic materials is solved. The composite cylinder consists of two parts: an inner, solid, round cylinder (which is hereinafter referred to as inclusion) and an outer, hollow, round cylinder. The inclusion was initially twisted and stretched (or compressed) along the axis and fastened to a strainless, hollow, outer cylinder. It is assumed that the radius of the inclusion and the inner radius of the outer cylinder match exactly only after the inclusion is subjected to the proposed initial deformation. Both cylinders have the same length after the initial deformation of the inner cylinder. The two cylinders are assumed perfectly bonded at that intermediate deformation stage (no slippage is allowed at their interface). Then the composite cylinder is twisted and stretched (or compressed) along the axis.

An exact analytical solution of this problem is obtained for arbitrary, nonlinear-elastic, isotropic, incompressible materials. This solution is based on the universal solution for incompressible materials – solving the problem of torsion, stretching, and changing the diameter of the round cylinder (Rivlin, 1949; Green and Adkins, 1960; Truesdell, 1972; Shield, 1982; Lurie, 1990; Saccomandi, 2001; Horgan and Murphy, 2011). The detailed investigation of the obtained solution is made for the case in which the cylinders are composed of the Mooney-type materials (Mooney, 1940).

The problem statement within the theory of repeatedly superimposed large strains can be formulated in the following way. Initially there are no stresses and strains in the inner cylinder. First, this cylinder is deformed – twisted, stretched along the axis or compressed – which involves changing the cylinder's diameter due to the material's incompressibility. This deformation results in the intermediate state of the cylinder, where it is in equilibrium. Then this cylinder is put into a hollow, non-deformed cylinder with an inner radius matching the first (solid) cylinder radius. The cylinders are fastened and additionally deformed – twisted, stretched along the axis or compressed – which involves changing their diameters as in the previous case. The composite cylinder

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passes to the final state, where it is in equilibrium. A lateral surface of the composite cylinder in this state is considered free of loads, and an axial force and a twisting moment are applied to its bases (Fig. 1).

The solution to this problem may be used for the analysis of coated fibers (Tsukrov and Drach, 2010) and allows us, in particular, to investigate the Poynting effect, which consists of changes to a cylindric body's length while torsion (Poynting, 1909). The research on this effect for a homogeneous cylinder composed of an incompressible material can be found in Zubov (2001).

Note that a number of problems of torsion of cylindrical or prismatic bodies at the large strains is solved in Zubov and Bogachkova (1995). The research of buckling of a hollow, nonlinear-elastic cylinder while it is under torsion, stretching along the axis and changing its diameter, can be found in Zubov and Sheidakov (2008). The extension and torsion of a *compressible* elastic circular cylinder is analyzed in Kirkinis and Ogden (2002). The combination of radial dilatation, helical shear and torsion of a hollow cylindrical tube made of an incompressible isotropic nonlinearly elastic material is investigated in De Pascalis et al. (2007).

Saravanan and Rajagopal (2003, 2005) investigated the inflation, tension, torsion, and shearing of an inhomogeneous, nonlinear-elastic, hollow circular cylinder under large strains. The preliminary strain of a part of the cylinder is not considered in those papers. It is assumed that the material properties vary along the radial direction. The semi-inverse method is used to solve the problems. Saravanan and Rajagopal (2003) solved the problems for *incompressible*, isotropic materials. The solution in quadratures is obtained. Saravanan and Rajagopal (2005) solved the problems for *compressible*, isotropic materials. The problems are reduced to the initial value problems for ordinary differential equations that are solved numerically. It is shown that the inhomogeneity of the material has a significant influence on the state of strains and stresses in the cylinder.

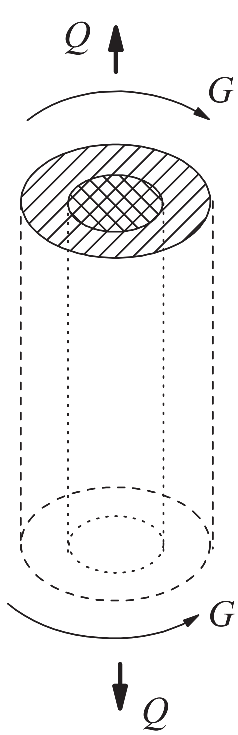

Fig. 1. Loading of a composite cylinder. An axial force Q and a twisting moment G are applied to the bases of the cylinder.

2. Nomenclature

We use the notation that is typical for the theory of superimposed large strains (Levin, 1998):

$\boldsymbol{\sigma}$	Cauchy stress tensor
$\mathbf{B}$	the Finger strain measure
$\mathbf{F}$	total deformation gradient
$\mathbf{F}_{\mathrm{init}}$	initial deformation gradient according to the transition from the initial state to the intermediate state
$\mathbf{F}_{\mathrm{add}}$	additional deformation gradient i.e., the deformation at the transition from the intermediate state to the final state
$\mathbf{E}$	identity tensor
$\mathrm{tr}\,\mathbf{A} = \mathbf{A}\cdot\cdot\mathbf{E}$	the trace of a second-rank tensor $\mathbf{A}$
$\mathbf{R}$	the particle radius vector at the final state
x_s $(s = 1, 2, 3)$	Cartesian coordinates of the particles at the initial (non-deformed) state
x'_m $(m = 1, 2, 3)$	coordinates at the intermediate state
X_k $(k = 1, 2, 3)$	coordinates at the final state
$\mathbf{i}_k$	coordinate orts of Cartesian coordinates
p	pressure in the incompressible body that is not expressed through deformation
ρ, θ, ζ	cylindrical coordinates of the particles of the inner cylinder in a natural (non-deformed) state
r, φ, z	cylindrical coordinates of the particles of the inner cylinder at the intermediate (pre-stressed) state and of the particles of the outer cylinder in a natural (non-deformed) state
R, Φ, Z	cylindrical coordinates of the cylinder particles at the final state

The aforementioned coordinates are related with the Cartesian coordinates as follows:

$$x_1 = \rho\cos\theta, \quad x_2 = \rho\sin\theta, \quad x_3 = \zeta,$$

$$x'_1 = r\cos\varphi, \quad x'_2 = r\sin\varphi, \quad x'_3 = z,$$

$$X_1 = R\cos\Phi, \quad X_2 = R\sin\Phi, \quad X_3 = Z.$$

Here, $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, $\mathbf{e}_\zeta$, $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$, $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ are the orts tangent to the coordinate lines of cylindrical coordinates:

$$\mathbf{e}_\rho = \mathbf{i}_1\cos\theta + \mathbf{i}_2\sin\theta, \quad \mathbf{e}_\theta = -\mathbf{i}_1\sin\theta + \mathbf{i}_2\cos\theta, \quad \mathbf{e}_\zeta = \mathbf{i}_3,$$

$$\mathbf{e}_r = \mathbf{i}_1\cos\varphi + \mathbf{i}_2\sin\varphi, \quad \mathbf{e}_\varphi = -\mathbf{i}_1\sin\varphi + \mathbf{i}_2\cos\varphi, \quad \mathbf{e}_z = \mathbf{i}_3,$$

$$\mathbf{e}_R = \mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi, \quad \mathbf{e}_\Phi = -\mathbf{i}_1\sin\Phi + \mathbf{i}_2\cos\Phi, \quad \mathbf{e}_Z = \mathbf{i}_3.$$

3. Initial strain of the inner cylinder

The initial (preliminary) strain of the inclusion is an axial stretching-compressing and torsion and is set by the following mapping (Rivlin, 1949; Green and Adkins, 1960; Truesdell, 1972; Lurie, 1990)

$$r = r(\rho), \quad \varphi = \theta + \psi_0\zeta, \quad z = \lambda_0\zeta, \tag{1}$$

$$\psi_0 = \mathrm{const}, \quad \lambda_0 = \mathrm{const}.$$

The tensor $\mathbf{F}_{\mathrm{init}}$ corresponding to the initial strain (1) is expressed in the following way:

$$\mathbf{F}_{\mathrm{init}} = \frac{dr}{d\rho}\mathbf{e}_r\otimes\mathbf{e}_\rho + \frac{r}{\rho}\mathbf{e}_\varphi\otimes\mathbf{e}_\theta + \psi_0 r\mathbf{e}_\varphi\otimes\mathbf{e}_\zeta + \lambda_0\mathbf{e}_z\otimes\mathbf{e}_\zeta. \tag{2}$$