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A semi-analytical elastic solution for stress field of lined non-circular tunnels at great depth using complex variable method

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ABSTRACT

In this paper, a semi-analytical elastic plane strain solution was provided for stress field around a lined non-circular tunnel subjected to uniform ground load. Concrete lining and the surrounding rock mass were assumed as linearly elastic materials. Due to complexity of the problem for non-circular geometric configurations, complex variable method introduced by Muskhelishvili and conformal mapping functions were used to determine stress components within concrete lining and the surrounding rock mass. Finally, the solution was validated by ABAQUS finite element software through an example. Very good agreement was demonstrated between semi-analytical and numerical solution although some discrepancies were found at tunnel corners where large curvature existed. It was demonstrated that the solution predicted stress components more accurately around the tunnels, especially the corners with large stress concentration. Practical significance of the solution was placed in the fact that it could be used as a quick-solver with high accuracy.

1. Introduction

Tunnels are main infrastructures in underground mining which are broadly used for transportation, water passage and other purposes such as underground cables. Liner is the primary support adopted to endure ground pressure. It has been of high interest for determining stress distribution within liner and rock medium with high level of accuracy. This subject has been studied by numerous researchers (Bobet, 2001, 2003; Einstein and Schwartz, 1979; Lee and Nam, 2001; Penzien and Wu, 1998; Savin, 1961; Timoshenko and Goodier, 2011).

Many studies have been carried out to determine stress and deformation field for tunnels by applying numerical methods. Exact solutions have been generally used to validate results of numerical methods. They are also largely employed in preliminary stages of design in order to find stress and deformation state. Furthermore, analytical solutions give an understanding of how final solutions are influenced by different parameters.

One of the useful analytical approaches implemented in two-dimensional elastic theory is Muskhelishvili's complex variable method (Muskhelishvili and Radok, 1953). The method investigated based on complex potential functions and conformal mapping method can determine stress components and deformations within the materials. Based on this method, Exadaktylos et al. proposed a closed-form solution for stress and displacement of semicircular tunnels as well as a semi-analytical solution for notched circular openings in homogenous elastic media (Exadaktylos and Stavropoulou, 2002; Exadaktylos et al., 2003). A similar method was proposed for stress and displacement around a circular tunnel in an elastic half-plane (Strack and Verruijt, 2002; Verruijt, 1997). However, the interaction between support and geomaterial has been rarely considered in the above-mentioned literature.

In case the problem is considered without reinforcement layer, the solution would come up much easier. When liner is included, two different regions should be assumed which increases complexity of the problem (England, 2009). For openings with simple geometry like ellipse or regular polygonal holes, a closed form solution could be extracted. However, when the opening and consequently reinforced layer has a complex geometry, a semi-analytical method is resulted, in which complex potential and conformal mapping functions are defined as power series.

An attempt was made in this study to find stress field of a lined non-circular tunnel at a great depth with a complex geometry configuration by applying Muskhelishvili's complex variable method. Rock mass and lining concrete were assumed as isotropic linear elastic materials.

2. General consideration

The problem involves a lined non-circular tunnel at depth H within linearly elastic geomaterial. The tunnel is located at such

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a great depth compared with the tunnel dimension that the problem is considered a reinforced hole in an infinite plane, subjected to a uniform stress state at infinity. Concrete and rock mass are assumed as isotropic and homogenous materials. It is supposed that liner is installed without any delay after tunnel excavation. The infinite plate on complex plane is divided into two isotropic homogenous regions of S_1 and S_2 bounded by contours L_1 and L_2 (Fig. 1). The regions S_1 and S_2 refer to rock mass and concrete lining with Young modulus E_1, E_2 and Poisson ratio ν_1, ν_2, respectively. Let $w(\xi)$ be a conformal mapping function which maps boundaries L_1 and L_2 into two concentric circles η_1 and η_2 with unit and R_0 radii. It is assumed that the conformal mapping function has the following expansion:

$$w(\xi) = Re^{i\frac{\pi}{2}}\left(\xi + \sum_{i=1}^{n}\omega_i\xi^{-i}\right) \quad (1)$$

Where $i = \sqrt{-1}$, R is a real constant affecting scale of the hole and $\frac{\pi}{2}$ is the angle by which the shape is rotated from its original position. The function $w(\xi)$ thoroughly maps each exterior point of unit circle η_1 into region S_1 and each point in the region γ_2 bounded by circles η_1 and η_2 into region S_2. According to Muskhilishvili and Kolosove's method, there are complex potential functions φ_1, ψ_1 and φ_2, ψ_2 which are analytic on regions S_1 and S_2, respectively. Stress components are determined based on these complex potential functions as follows:

$$\left.\begin{aligned}&\sigma_\rho + \sigma_\theta = 2\left(\frac{\varphi_j'(\xi)}{w'(\xi)} + \overline{\frac{\varphi_j'(\xi)}{w'(\xi)}}\right)\\ &\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = \frac{2e^{2i\theta}}{\overline{w'(\xi)}}\left\{\overline{w(\xi)}\,\frac{\varphi_j''(\xi)w'(\xi) - \varphi_j'(\xi)w''(\xi)}{(w'(\xi))^2} + \psi_j'(\xi)\right\}\end{aligned}\right\}, \quad j = 1,2 \quad (2)$$

where σ_ρ, σ_θ and $\tau_{\rho\theta}$ are radial, circumferential and tangential stress components, respectively. The complex functions φ_1 and ψ_1 defined on region γ_1 are

$$\varphi_1(\xi) = \Gamma w(\xi) + \varphi_0(\xi), \quad \psi_1(\xi) = \Gamma' w(\xi) + \psi_0(\xi) \quad (3)$$

where

$$\varphi_0(\xi) = \sum_{j=0}^{\infty} a_j\xi^{-j}, \quad \psi_0(\xi) = \sum_{j=0}^{\infty} b_j\xi^{-j} \quad (4)$$

$\varphi_0(\xi)$ and $\psi_0(\xi)$ are holomorphic functions with $\varphi_0(\infty) = 0$ and $\psi_0(\infty) = 0$, Γ and Γ' are real and complex constants with regard to stress state at infinity, which are determined as follows:

$$\begin{aligned}\Gamma &= \tfrac{1}{4}(\sigma_1 + \sigma_2) = \tfrac{1+K}{4}\gamma H\\ \Gamma' &= -\tfrac{1}{2}(\sigma_1 - \sigma_2)e^{-2i\alpha} = -\tfrac{1-K}{2}\gamma H\end{aligned} \quad (5)$$

where σ_1 and σ_2 are principal stress components at infinity, α is the angel made between σ_1 direction and x axis and K and γ are lateral coefficient pressure and stress gradient of rock mass, respectively.

The complex functions φ_2 and ψ_2 defined on region γ_2 are determined as follows:

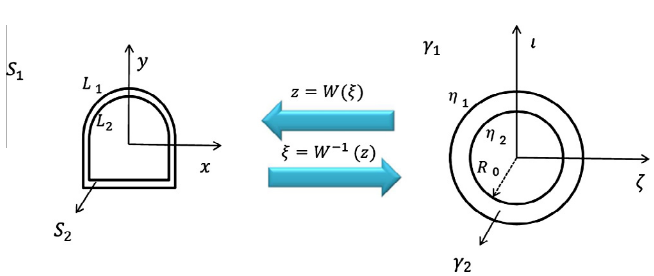

Fig. 1. Conformal mapping of tunnel cross-section in z-plane into two concentric circles in ξ-plane.

$$\varphi_2(\xi) = R_1(\xi) + R_2(\xi), \quad \psi_2(\xi) = Q_1(\xi) + Q_2(\xi) \quad (6)$$

where R_1, R_2 and Q_1, Q_2 are assumed analytic functions on region γ_2 which have the following series expansions:

$$Q_1(\xi) = \sum_{j=1}^{\infty} e_j\xi^{-j}, \quad R_1(\xi) = \sum_{j=1}^{\infty} c_j\xi^{-j} \quad (7)$$

$$Q_2(\xi) = \sum_{j=0}^{\infty} f_j\xi^{j}, \quad R_2(\xi) = \sum_{j=0}^{\infty} d_j\xi^{j} \quad (8)$$

Stress functions φ_1, ψ_1 and φ_2, ψ_2 should satisfy boundary conditions on η_1 and η_2 circles:

$$\begin{aligned}&\frac{k_1}{G_1}\varphi_0(t) - \frac{1}{G_1}\left(\frac{w(t)}{\overline{w'(t)}}\overline{\varphi_0'(t)} + \overline{\psi_0(t)}\right)\\ &= \frac{k_2}{G_2}\varphi_2(t) - \frac{1}{G_2}\left(\frac{w(t)}{\overline{w'(t)}}\overline{\varphi_2'(t)} + \overline{\psi_2(t)}\right) \quad \text{on } \eta_1\end{aligned} \quad (9)$$

$$\varphi_1(t) + \frac{w(t)}{\overline{w'(t)}}\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = \varphi_2(t) + \frac{w(t)}{\overline{w'(t)}}\overline{\varphi_2'(t)} + \overline{\psi_2(t)} \quad \text{on } \eta_1 \quad (10)$$

$$\varphi_2(t) + \frac{w(t)}{\overline{w'(t)}}\overline{\varphi_2'(t)} + \overline{\psi_2(t)} = 0 \quad \text{on } \eta_2 \quad (11)$$

where $G_i = \frac{E_i}{2(1+\nu_i)}$, $k_i = \begin{cases}3 - 4\nu_i & \text{plane strain}\\ \frac{3-\nu_i}{1+\nu_i} & \text{plane stress}\end{cases}$ $(i = 1,2)$.

Eqs. (9) and (10) are concerned with continuity of deformation and stress field across the rock-concrete interface due to no-slip condition (i.e., there is a perfect adherence between liner and rock mass so that they have common deformation along the interface) and Eq. (11) is about boundary condition along the observation hole (which involves zero traction along the boundary L_2). It should be considered that expressions $\Gamma w(\xi)$ and $\Gamma' w(\xi)$ are not incorporated into continuity Eq. (9) since they define pre-stress and pre-deformation field in rock mass when tunnels are excavated.

3. Solution

To acquire a solution for stress functions φ_1, ψ_1 and φ_2, ψ_2, at first, Eqs. (9)–(11) would be conjugated considering the fact that $\bar{t} = \frac{1}{t}$ for each point on η_1:

$$\frac{k_1}{G_1}\bar{\varphi}_0\left(\frac{1}{t}\right) - \frac{1}{G_1}\left(\frac{\overline{w}(\frac{1}{t})}{w'(t)}\varphi_0'(t) + \psi_0(t)\right) = \frac{k_2}{G_2}\bar{\varphi}_2\left(\frac{1}{t}\right) - \frac{1}{G_2}\left(\frac{\overline{w}(\frac{1}{t})}{w'(t)}\varphi_2'(t) + \psi_2(t)\right) \quad (12)$$

$$\bar{\varphi}_1\left(\frac{1}{t}\right) + \frac{\overline{w}(\frac{1}{t})}{w'(t)}\varphi_1'(t) + \psi_1(t) = \bar{\varphi}_2\left(\frac{1}{t}\right) + \frac{\overline{w}(\frac{1}{t})}{w'(t)}\varphi_2'(t) + \psi_2(t) \quad (13)$$

$$\bar{\varphi}_2\left(\frac{R_0}{t}\right) + \frac{\overline{w}(\frac{R_0}{t})}{w'(R_0t)}\varphi_2'(R_0t) + \psi_2(R_0t) = 0 \quad (14)$$

By multiplying Eqs. (12)–(14) by $\frac{dt}{t-\xi}$ and integrating it along unit circle η_1 for $|\xi| > 1$ and $|\xi| < 1$, a set of 6 equations is obtained which include 6 unknown complex functions.for $|\xi| \succ 1$:

$$\begin{aligned}&-\frac{1}{G_1}\cdot\frac{1}{2\pi i}\int_{\eta_1}\frac{\overline{w}(\frac{1}{t})}{w'(t)}\varphi_0'(t)\frac{dt}{t-\xi} + \frac{1}{G_1}\psi_0'(\xi) - \frac{1}{G_1}b_0\\ &= -\frac{k_2}{G_2}\overline{R}_2\left(\frac{1}{\xi}\right) + \frac{k_2}{G_2}d_0 - \frac{1}{G_2}\cdot\frac{1}{2\pi i}\int_{\eta_1}\frac{\overline{w}(\frac{1}{t})}{w'(t)}R_1'(t)\frac{dt}{t-\xi}\\ &\quad - \frac{1}{G_2}\cdot\frac{1}{2\pi i}\int_{\eta_1}\frac{\overline{w}(\frac{1}{t})}{w'(t)}R_2'(t)\frac{dt}{t-\xi} + \frac{1}{G_2}Q_1(\xi)\end{aligned} \quad (15)$$