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A physical perspective of the length scales in gradient elasticity through the prism of wave dispersion

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A R T I C L E   I N F O

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A B S T R A C T

The goal of this study is to understand the physical meaning and evaluate the intrinsic length scale parameters, featured in the theories of gradient elasticity, by deploying the analytical treatment and experimental measurements of the dispersion of elastic waves. The developments are focused on examining the propagation of longitudinal waves in an aluminum rod with periodically varying cross-section. First, the analytical solution for the dispersion relationship, based on the periodic cell analysis of a bi-layered laminate and Bloch theorem, is compared to two competing models of gradient elasticity. It is shown that the customary gradient elastic model with two length-scale parameters is able to capture the dispersion accurately up to the beginning of the first band gap. On the other hand, the gradient elastic model with an additional length scale (affiliated with the fourth-order time derivative in the field equation) is shown to capture not only the first dispersion branch before the band gap, but also the band gap itself and the preponderance of the second branch. Closed form relations between the microstructure parameters and the intrinsic length scales are obtained for both gradient elasticity models. By way of the asymptotic treatment in the limit of a weak contrast between the laminae, a clear physical meaning and scaling of the length-scale parameters was established in terms of: (i) the microstructure (given by the size of the unit cell and the contrast between the laminae), and (ii) thus induced dispersion relationship (characterized by the location and the width of the band gap). The analysis is verified through an experimental observation of wave dispersion, and wave attenuation within the band gap. A comparison between the analytical treatment, the gradient elastic model with three intrinsic length scales, and experimental measurements demonstrates a good agreement over the range of frequencies considered.

1. Introduction

It is well known that the classical approaches in continuum mechanics have limitations in mimicking the behavior of materials with microstructure. For instance, the conventional theory of linear elasticity cannot capture the dispersion characteristics of body waves propagating through a material that appears to be homogeneous at the meso scale. This and other limitations provide a motivation towards developing enriched continuum models that are able to capture the size effects by introducing intrinsic length scales synthesizing the key features of the sub-scale material structure. In the context of linear elasticity, the so-called gradient elasticity models have been considered for almost half a century. A brief survey of the theory of gradient elasticity and its applications are outlined in the sequel; for a comprehensive overview of the available formulations, in terms of both static and dynamic problems, the reader is referred to Askes and Aifantis (2011).

The general theory of gradient elasticity was established in the 1960s by Toupin (1962, 1964) and Mindlin (1964). However, the problem with applying the general theory resides in a large number of intrinsic parameters, which makes the experimental measurements thereof extremely difficult. For this reason, a multitude of reduced models with a manageable number of length-scale parameters have been proposed. For instance, static theories of gradient elasticity with a single length-scale parameter are typically used to deal with the stress singularities near the crack tip (Aifantis, 1992; Gourgiotis and Georgiadis, 2009). Similar approaches aiming to mitigate the singularity at the dislocation core can be found for example in Gutkin and Aifantis (1999). Another area where the models of gradient elasticity are found to be useful is the prediction of the wave dispersion characteristics in heterogeneous or discrete systems, see e.g. (Mindlin, 1964; Muhlhaus and Oka, 1996; Gonella et al., 2011). A variety of dynamic models, that account for higher-order inertial terms and thus allow for a more detailed description of the wave dispersion, have been considered in Metrikine and Askes (2002), Askes and Metrikine (2002) and Askes and Aifantis (2009). An in-depth discussion of the dispersion phenomena brought about by the models of gradient elasticity can

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be found in Papargyri-Beskou et al. (2009) and Fafalis et al. (2012). Many such models, however, are found to be non-causal in the context of wave propagation; to deal with the causality issue, one option is to include the fourth-order time derivative (and affiliated length-scale parameter) in the formulation (Metrikine, 2006). As shown in Pichugin et al. (2008), such consideration of the fourth-order time derivative also caters for an elevated asymptotic accuracy of the models of gradient elasticity within the framework of discrete models. A detailed comparison between the dispersive characteristics of various simplified models of gradient elasticity can be found in Askes et al. (2008).

One of the biggest challenges in dealing with the theories of gradient elasticity is a physical interpretation of the featured length-scale parameters in terms of given microstructure (Askes and Aifantis, 2011). When tackling the materials with randomly distributed heterogeneities, a numerical homogenization over the representative volume element (RVE) is typically utilized (Kouznetsova et al., 2002; Kouznetsova et al., 2004; Gitman et al., 2007). In this situation, the parameters of gradient elasticity are related to the size of the RVE, although such computational platform brings little to no physical clarity when dealing with multiple length scales. Another possibility is to derive the latter from the expansion of a discrete model (Metrikine and Askes, 2002; Metrikine, 2006), described as a specific arrangement of masses and springs. One drawback of this approach, however, is the lack of a precise relationship between the discrete model and the structure of a given heterogeneous continuum. As an alternative to the foregoing treatments, one-dimensional multi-scale homogenization of a bi-layered laminate was considered in Chen and Fish (2001) and Fish et al. (2002). Such approach allows one to obtain a closed-form expression for the intrinsic length scale in terms of the parameters of the laminate. In the present investigation, it is shown that the homogenization strategy in Chen and Fish (2001) and Fish et al. (2002) amounts to a Taylor series expansion of the affiliated dispersion relationship, enabling the gradient elasticity model to capture the initial slope and the initial curvature thereof. In this case, however, the theory of gradient elasticity fails to capture the salient "meso"-frequency features brought about by the microstructure, such as the presence of the band gaps.

An experimental measurement of the length-scale parameters of gradient elasticity is likewise an arduous task, as their effect on the sensory data may be limited. For example, static methods were deployed in Aifantis (1999), Lam et al. (2003) and Askes et al. (2012), where the size effects in torsion, bending and fracture were used to determine the relevant length-scale parameters. In the context of fracture mechanics, the intrinsic length scale may be calibrated assuming the equality between the maximum principal stress and the uniaxial tensile strength of a material (Askes et al., 2012). On the other hand, the dynamic methods geared toward exposing the intrinsic length scales typically entail measurements of wave dispersion. For instance, the ultrasound wave dispersion in polycrystalline metals was studied in Savin et al. (1970), while Jakata and Every (2008) deployed neutron scattering to obtain the dispersion characteristics of cubic crystals. In Wang and Hu (2005), on the other hand, the dispersion of flexural waves in carbon nanotubes (obtained via molecular dynamic simulations) was compared to that stemming from the theories of gradient elasticity. In most wave-based techniques, however, the attention is focused on the low-frequency approximation of the germane dispersion relationship, whereby the dispersion is treated as a small correction to the "baseline" non-dispersive wave model. Unfortunately, such paradigm does not cater for distinguishing between multiple length scales of gradient elasticity, e.g. between the "static" length scales and those that are inertia-related.

To help bridge the gap, this work employs both theoretical analysis and experimental observations to shed light on the fundamental relationship between the material microstructure, wave dispersion, and equivalent-homogeneous parameters of gradient elasticity. The primary goals are to understand the physical meaning of the length-scale parameters featured by two prominent models of gradient elasticity and to attempt their measurement in an experimental setting. To facilitate the analytical treatment of the dispersion analysis, a one-dimensional problem of longitudinal wave propagation in an elastic rod with periodically varying cross-section is adopted as a modeling platform.

2. Problem statement

Consider the propagation of longitudinal waves in a non-uniform rod characterized by the periodic pattern of rectangular cuts as shown in Fig. 1. The cuts, also referred to as the "damage", endow the wave propagation problem with a length scale L (the length of the unit cell) and a dimensionless parameter $\gamma = A_2/A_1$, which describes the ratio between the damaged and intact cross-sectional areas. The analysis of the wave dispersion in such periodic system is performed from both theoretical and experimental perspectives, with the aim of establishing a link between the parameters of gradient elasticity and germane material microstructure.

3. Wave dispersion in a periodic bi-layered structure

To study dispersion of elastic waves in a homogeneous rod with varying cross-section, it is useful to first consider an equivalent one-dimensional (1D) model that assumes constant cross-section but varying material properties as shown in the right panel of Fig. 1. Assuming time-harmonic excitation at frequency ω, the governing equation for the propagation of longitudinal waves in the latter system reads

$$\frac{\partial}{\partial x}\left(E(x)\frac{\partial u}{\partial x}\right) + \rho(x)\omega^2 u = 0, \tag{1}$$

where x is the axial coordinate; u carries the implicit time factor $e^{i\omega t}$, and $\rho(x)$ and $E(x)$ signify respectively the (varying) mass density and Young's modulus of the rod. In this setting, the analysis of wave propagation through material with periodic structure can be performed using the Bloch analysis e.g. (Brillouin, 1946), which can be formulated for the unit cell as

$$\begin{bmatrix} u \\ \sigma \end{bmatrix}\Bigg|_{x=L} = e^{-ikL}\begin{bmatrix} u \\ \sigma \end{bmatrix}\Bigg|_{x=0}, \tag{2}$$

where u and $\sigma = E\,\partial u/\partial x$ denote respectively the axial displacement and affiliated normal stress, and k is the wave number. Relationship (2) can be understood as the boundary condition for the problem of wave propagation through a bi-layered material that is governed by (1). With the aid of the transfer matrix approach, on the other hand, the solution of (1) can be written as

$$\begin{bmatrix} u \\ \sigma \end{bmatrix}\Bigg|_{x=L} = \boldsymbol{T}_2\boldsymbol{T}_1\begin{bmatrix} u \\ \sigma \end{bmatrix}\Bigg|_{x=0}, \tag{3}$$

where the matrices $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are given respectively by

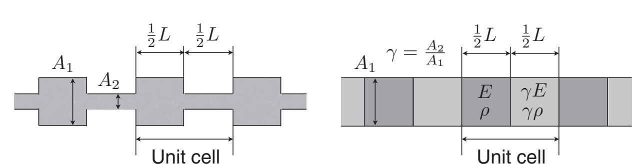

Fig. 1. Schematics of the elastic rod with rectangular cuts (left) and its 1D approximation (right).