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On the use of universal relations in the modeling of transversely isotropic materials

Edvige Pucci, Giuseppe Saccomandi *

Dipartimento di Ingegneria Industriale, Università degli Studi di Perugia, 06125 Perugia, Italy

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ABSTRACT

A material is of *coaxial type* if the Cauchy stress tensor $\boldsymbol{T}$ and the strain tensor $\boldsymbol{B}$ are coaxial for all deformations. Clearly a hyperelastic material is of coaxial type if and only if it is isotropic. Here we present a weaker definition of materials of coaxial type. Anisotropic materials may be of a coaxial type in a weak sense if for a given *specific* $\boldsymbol{B}$ we have that $\boldsymbol{TB} = \boldsymbol{BT}$. We denote these materials $\boldsymbol{B}$-coaxial. We show that for transverse isotropic materials weak coaxial constitutive equations may be characterized using universal relations. We discuss the impact of $\boldsymbol{B}$-coaxial materials in the modeling of soft tissues. We conclude that $\boldsymbol{B}$-coaxial materials are a strong evidence that in real world materials two anisotropic invariants are always necessary to model in a meaningful and correct way single fiber reinforced materials.

1. Introduction

In recent times there has been a huge interest in the modeling of the mechanical response of incompressible, transversely isotropic, nonlinearly elastic materials and this because there are many examples of biological, soft tissue reinforced with bundles of fibres that have an approximate single preferred direction.

If we assume soft tissue non-dissipative, as is commonly the case, then its mechanical response is determined completely by the strain-energy function. We consider a compressible elastic solid with a single fiber direction identified by the unit vector $\boldsymbol{M}$ in the reference configuration. The strain-energy function W (defined per unit reference volume) depends on five independent invariants, usually denoted I_1, I_2, I_3, I_4, I_5 (see, for example, Spencer, 1972). Here I_1, I_2, and I_3 are the principal invariants of the right Cauchy–Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ (equivalently of the left Cauchy–Green deformation tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$), $\boldsymbol{F}$ being the deformation gradient tensor relative to the (unstressed) reference configuration. Thus,

$$I_1 = \mathrm{tr}\boldsymbol{C}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)\right], \quad I_3 = \det\boldsymbol{C}. \tag{1.1}$$

The invariants associated with the fiber reinforcement are defined, introducing the notation $\boldsymbol{m} = \boldsymbol{FM}$, as

$$I_4 = \boldsymbol{m}\cdot\boldsymbol{m} = \boldsymbol{M}\cdot\boldsymbol{CM}, \quad I_5 = \boldsymbol{m}\cdot\boldsymbol{Bm} = \boldsymbol{M}\cdot\boldsymbol{C}^2\boldsymbol{M}. \tag{1.2}$$

The Cauchy stress tensor, $\boldsymbol{T}$, is given by the standard formula

$$\boldsymbol{T} = J^{-1}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{F}}, \tag{1.3}$$

where $J = \det\boldsymbol{F} = I_3^{1/2}$. Therefore we have

$$J\boldsymbol{T} = 2W_1\boldsymbol{B} + 2W_2(I_1\boldsymbol{I} - \boldsymbol{B})\boldsymbol{B} + 2I_3W_3\boldsymbol{I} + 2W_4\boldsymbol{m}\otimes\boldsymbol{m} + 2W_5(\boldsymbol{m}\otimes\boldsymbol{Bm} + \boldsymbol{Bm}\otimes\boldsymbol{m}), \tag{1.4}$$

where $\boldsymbol{I}$ is the identity tensor and $W_i = \partial W/\partial I_i$ $(i = 1, \ldots, 5)$

It is interesting to notice that in modeling incompressible transversely isotropic materials, it is usual to ignore the I_2, I_5 invariants and to adopt the assumption that $W = W(I_1, I_4)$ see for example, among many others, Horgan and Murphy (2012), Humphrey and Yin (1987) and Humphrey et al. (1990). In a recent series of papers this constitutive assumption has been deeply criticized, see for example Destrade et al. (2013) and Vergori et al. (2013) and the *unphysical* features of the constitutive assumption $W = W(I_1, I_4)$ have been stressed out into details.

The aim of this note is to corroborate the above mentioned results showing that the problems we encounter in modeling transverse isotropic materials are not related to the choice of the invariant I_4 but to the fact that we need to incorporate in the constitutive model, in a independent way, both the I_4 and I_5 invariants. If this does not happens, because for example we use only a single anisotropic invariant (for an example a combination of I_4 and I_5) it is possible to encounter serious mechanical problems.

Our arguments are based on a remarkable use of universal relations (Saccomandi, 2001). An universal relation is an equation that holds for every material in a specified class. For compressible and incompressible isotropic materials, universal relations are

* Corresponding author. Tel.: +39 0755853707.
E-mail addresses: edvige.pucci@unipg.it (E. Pucci), giuseppe.saccomandi@unipg.it (G. Saccomandi).

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mainly generated by the coaxiality of strain and stress and they were first discovered by Rivlin (1997) for the simple shear deformation and subsequently examined in detail for a huge class of deformations by Beatty (1987). For transverse isotropic materials universal relations have been obtained by Saccomandi and Vianello (1997).

Vianello (1996) has found that the coaxiality between stress and strain (and therefore universal relations) is strictly related to a classical question in applied mechanics of composite materials:

How can we determine, for a given body and a given deformation, the conditions under which an appropriate rotation yields a critical value (in particular, a maximum or a minimum) for the stored energy?

Considering this optimization problem in the framework of transverse isotropic materials by using universal relations we provide an example of transverse isotropic materials, depending only on a single anisotropic invariant, that behaves in a very *special* way in a important class of deformations. These materials have been recently proposed as a feasible constitutive model for soft tissues, but our findings cast a strange light on the possibility to use such models in real world applications.

2. Basic results

Vianello (1996) has given an elegant answer to the optimization problem that we have posed in the Introduction. The solution proposed by Vianello is valid in linear and non-linear elasticity:

Rotations which are critical for the stored energy are exactly those that correspond to coaxial states of stress and strain.

To be more precise let us fix a deformation, i.e. a gradient of deformation $\boldsymbol{F}$. Now let us rotate the body before the deformation is applied the new deformation gradient is given by $\boldsymbol{FQ}$, where $\boldsymbol{Q}$ is the rotation. For an isotropic body we have that $W(\boldsymbol{F}) = W(\boldsymbol{FQ})$ because the symmetry group of an isotropic material is the full rotation group, says *Rot*. Indeed, for an isotropic material is always $\boldsymbol{TB} = \boldsymbol{BT}$. On the other hand, for an anisotropic body, $W(\boldsymbol{F})$ is, in general, different from $W(\boldsymbol{FQ})$.

Therefore is meaningful to consider, for a given $\boldsymbol{F}$, the function defined as follow

$$\boldsymbol{Q} \to \Sigma(\boldsymbol{Q}) := W(\boldsymbol{FQ}), \quad \forall \boldsymbol{Q} \in Rot. \tag{2.1}$$

Being Σ a continuous function over a compact set, in view of the classical Weierstrass theorem it follows that this function has an absolute minimum and an absolute maximum.

Vianello (1996) has shown that a given rotation Q is critical for Σ if and only if the corresponding $\boldsymbol{T}$ and $\boldsymbol{B}$ are coaxial. This means that for anisotropic materials we have to expect that the value Σ is not constant for all $\boldsymbol{Q} \in Rot$ and, if this happens, this is because we are in a highly *degenerate* and special case.

Moreover, Vianello (1996) defines that a material is of *coaxial type* if the stress tensor $\boldsymbol{T}$ and the strain tensor $\boldsymbol{B}$ are coaxial for all deformations. Clearly a hyperelastic material is of coaxial type if and only if it is isotropic.

On the other hand, we may have a *weaker* definition of a material of coaxial type. Anisotropic materials may be of a coaxial type in a weak sense if for a *given* $\boldsymbol{B}$ we have that $\boldsymbol{TB} = \boldsymbol{BT}$. Because this results depends on the specific $\boldsymbol{B}$, we denote these materials as $\boldsymbol{B}$-coaxial. It is possible to show that in the class of transverse isotropic materials we have a class of constitutive equations of a coaxial type in this weak sense and to this end we have to consider the mathematical tool of universal relations.

It is well known from a celebrated Beatty's paper (1987), that coaxiality between stress and strain is a *machine* to generate universal relations for isotropic materials. Roughly speaking this means that for isotropic materials if we fix a general deformation it is possible to have three universal relations[1] and these universal relations may be represented as the components of the axial vector associated with the skew tensor $\boldsymbol{TB} - \boldsymbol{BT}$.

Saccomandi and Vianello (1997), have shown that for transverse isotropic materials, in general, we have only one universal relation. This relation may be obtained considering that, from (1.4), it is

$$J(\boldsymbol{TB}-\boldsymbol{BT}) = 2W_4(\boldsymbol{m}\otimes\boldsymbol{Bm} - \boldsymbol{Bm}\otimes\boldsymbol{m}) + 2W_5(\boldsymbol{m}\otimes\boldsymbol{B}^2\boldsymbol{m} - \boldsymbol{B}^2\boldsymbol{m}\otimes\boldsymbol{m}). \tag{2.2}$$

Let us denote with $(\boldsymbol{W})_\times$ the axial vector associated to a given a skew tensor $\boldsymbol{W}$. Both sides in (2.2) represent a skew tensor and therefore it must be using basic vector algebra

$$J(\boldsymbol{TB} - \boldsymbol{BT})_\times = 2\boldsymbol{m} \times \left(W_4\boldsymbol{B} + W_5\boldsymbol{B}^2\right)\boldsymbol{m}. \tag{2.3}$$

From (2.3) it is clear that

$$(\boldsymbol{TB} - \boldsymbol{BT})_\times \cdot \boldsymbol{m} = 0. \tag{2.4}$$

The (2.4) is the only universal relation valid for *all* $\boldsymbol{B}$ and for *all* transverse isotropic material $W = W(I_1, I_2, I_3, I_4, I_5)$. Relation (2.4) is a *partial* coaxiality condition in the fiber direction.

2.1. Additional universal relations

2.1.1. Special deformations

Let us consider special deformations such that $\boldsymbol{m} \times \boldsymbol{Bm} \cdot \boldsymbol{B}^2\boldsymbol{m} = 0$, i.e. there exists two scalar δ_1 and δ_2 such that

$$\boldsymbol{B}^2\boldsymbol{m} = \delta_1\boldsymbol{m} + \delta_2\boldsymbol{Bm}. \tag{2.5}$$

In this case, it must be

$$(\boldsymbol{I} - \boldsymbol{m}\otimes\boldsymbol{m})\boldsymbol{Bm} = \delta_2^{-1}(\boldsymbol{I} - \boldsymbol{m}\otimes\boldsymbol{m})\boldsymbol{B}^2\boldsymbol{m}. \tag{2.6}$$

where δ_2^{-1} does not depends on $\boldsymbol{M}$.

When (2.5) holds we have the possibility to rewrite (2.3) as

$$J(\boldsymbol{TB} - \boldsymbol{BT})_\times = 2\boldsymbol{m} \times \left(W_4 + \delta_2^{-1}W_5\right)(\boldsymbol{I} - \boldsymbol{m}\otimes\boldsymbol{m})\boldsymbol{Bm} \tag{2.7}$$

and

$$(\boldsymbol{TB} - \boldsymbol{BT})_\times \cdot (\boldsymbol{I} - \boldsymbol{m}\otimes\boldsymbol{m})\boldsymbol{Bm} = 0 \tag{2.8}$$

is an additional universal relation valid for any transverse isotropic material $W = W(I_1, I_2, I_3, I_4, I_5)$.

In the highly *degenerate* case where the three vectors $\boldsymbol{m}, \boldsymbol{Bm}$ and $\boldsymbol{B}^2\boldsymbol{m}$ are parallel full coaxiality between stress and strain is recorded also for all transverse isotropic materials. For compressible materials this is the case of pure dilatation.

The condition (2.6) is not exotic. If we consider just a triaxial stretch

$$x = \lambda_1 X, \quad y = \lambda_2 Y, \quad z = \lambda_1 Z$$

and a generic arrangement of fibers in the reference configuration (i.e. a generic unit vector $\boldsymbol{M}$) there are infinite ways to arrange (2.6). For example, if $M_3 = 0$ the (2.6) is satisfied if

$$\lambda_1^2 M_1^2 + \lambda_2^2 M_2^2 = 1.$$

[1] The rigorous proof that in the general case for isotropic elastic materials we have three universal relation is contained in (Pucci and Saccomandi, 1997) where methods of algebraic geometry are used.