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A series solution for the effective properties of incompressible viscoelastic media

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ABSTRACT

This paper presents a series solution for the homogenization problem of a linear viscoelastic periodic incompressible composite. The method uses the Laplace transform and the correspondence principle which are combined with the classical expansion along Neumann series of the solution of the periodic elasticity problem in Fourier space. The terms of the Neumann series appear as decoupled, containing geometry dependent terms and viscoelastic properties dependent terms which are polynomial fractions whose inverse Laplace transforms are provided explicitly.

1. Introduction

The methods used for predicting the effective properties of heterogeneous viscoelastic composites comprise solutions to the problem of the complex moduli (Brinson and Lin, 1998) with applications to dynamic problems, but the most important practical problem is to predict relaxation or creep functions. This last objective is generally attained by using Laplace transform of the equations, the main problem being to produce accurately the inverse Laplace transform. This was effected for Mori–Tanaka or Self-Consistent modelings (Beurthey et al., 2000; Rougier et al., 1994; Le et al., 2007), which allow to obtain the relaxation function explicitly or by using a simple 1D integral. In addition, all effective behaviours must comply with some asymptotic conditions, as obtained for example in the case of Maxwell constituents (Suquet, 2012).

The determination of the effective properties of periodic media using the classical Neumann series was used from a theoretical point of view since a long time (Brown, 1955), for conductivity, or for elasticity, by using the related Green's tensors. The practical application in conduction and elasticity rests on iterative schemes and on the use of the Fourier transform because the Fourier transform of the Green's tensor is known explicitly for an homogeneous medium in the case of elastic constitutive equations (Michel et al., 1999, 2001; Monchiet and Bonnet, 2012; Moulinec and Suquet, 2003, 1994, 1998; Bonnet, 2007). Approximate solutions based on Nemat-Nasser et al. (1982), which use Fourier transforms of the solutions, can produce explicit results in the case of viscoelastic components (Luciano and Barbero, 1994; Barbero and Luciano, 1995; Hoang-Duc et al., 2013) but these solutions are no more valid for high concentrations of inclusions or high contrasts. Accurate solutions at any concentration were obtained either by time-step integration (Lahellec and Suquet, 2007) or by numerical Laplace inverse, generally using collocation methods (Yi et al., 1998). However, a method based on Fourier transform, but which does not need numerical time-step integration or numerical Laplace inversion would be highly desirable. This is the aim of the paper.

In the following, this method will be called "NS method". The solution for determining the macroscopic behavior of viscoelastic periodic media is developed by using the classical Neumann Series for the effective elastic properties.

The paper is organized as follows: The constitutive relation used for the individual constituents is presented in Section 2. Then, we present in Section 3 simplified formulations of the effective properties of composite elastic media made of isotropic constituents with a decoupling of elastic properties and geometry properties in each term of the Neumann series. This decoupling appears only in some specific cases, including the case of incompressible constituents. This result is used in the next section to determine the expression of the relaxation function of the viscoelastic periodic composites at the macroscale. Finally, the method is checked against results coming from previous works.

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2. Linear viscoelastic behavior

2.1. Constitutive equations for an isotropic viscoelastic medium

In the following, a composite material is studied where the constituting phases are either elastic or non ageing viscoelastic. The constitutive stress–strain relation of a non-ageing viscoelastic material is given classically (Christensen, 1969; Salençon, 2009), by a Stieltjes integral as:

$$\boldsymbol{\sigma}(t) = \int_0^t \mathbb{R}(t-\tau) : \frac{d\boldsymbol{\epsilon}(\tau)}{d\tau} d\tau = \mathbb{R}(: \circledast)\dot{\boldsymbol{\epsilon}} \tag{1}$$

or reversely:

$$\boldsymbol{\epsilon}(t) = \int_0^t \mathbb{J}(t-\tau) : \frac{d\boldsymbol{\sigma}(\tau)}{d\tau} d\tau = \mathbb{J}(: \circledast)\dot{\boldsymbol{\sigma}} \tag{2}$$

where $\mathbb{R},\ \mathbb{J}$ are tensorial relaxation and creep functions. The dot denotes the time derivative and the convolution of two functions f and g, denoted as "$f \circledast g$", is defined by:

$$(f \circledast g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt \tag{3}$$

For a viscoelastic isotropic material, tensor $\mathbb{R}$ depends only on two scalar functions $R_\kappa(t)$ and $R_\mu(t)$ which are relaxation functions for compression and shear. The behavior of the material can be expressed by using the following form:

$$\boldsymbol{\sigma}(t) = R_\kappa(t) \circledast tr\dot{\boldsymbol{\epsilon}}(t)\mathbf{1} + 2R_\mu(t) \circledast \dot{\mathbf{e}}(t) \tag{4}$$

where $\mathbf{e}$ is the deviator of the strain tensor.

The viscoelastic constitutive equations of an isotropic viscoelastic material are therefore defined by two relaxation functions: $R_\kappa(t)$ and $R_\mu(t)$.

2.2. Laplace–Carson transform

The Laplace–Carson transform $f^*(p)$ of a real function $f(t),\ t \geqslant 0$ is obtained from its Laplace transform $\tilde{f}(s)$ by:

$$f^*(s) = s\tilde{f}(s) = s\int_0^\infty e^{-st} f(t)dt \tag{5}$$

Effecting the Laplace–Carson transform of the first expression in (4) leads to:

$$\boldsymbol{\sigma}^*(s) = R_\kappa^*(s) tr\boldsymbol{\epsilon}^*(s)\mathbf{1} + 2R_\mu^*(s)\mathbf{e}^*(s) \tag{6}$$

where s is the Laplace variable.

These expressions show that for any fixed value of s, the stress–strain relation in Laplace–Carson space is formally equivalent to the elasticity constitutive equation of an isotropic elastic material. This constitutes the "correspondence principle".

3. Decoupled forms of the overall properties of elastic periodic composites in specific cases

The paper presents different forms of the overall properties under the form of a series whose all terms are decoupled into two parts: the first part depends only on the microstructure and the second part depends only on the elastic properties. Such a decoupling is possible only in specific cases. So, different cases of series comprising decoupled terms are presented: two different forms (strain formulation and stress formulation) in the case of incompressible media and the strain formulation for a specific case of composite containing compressible materials. An example of result obtained by this method is shown and the main results coming from the literature are presented concerning the convergence of the series.

3.1. Basic equations of the problem

Let us consider a periodic composite built on a periodic cell Ω as in Fig. 1 by translation along the three directions of the space.

One denotes by $2a_i$ $(i = 1,2,3)$, the dimension along direction x_i of a basic parallelepipedic cell. Then the displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the strain field $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{x})$ and the stress field $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$ induced by a macroscopic strain tensor $\mathbf{E}$ are solutions of:

$$\begin{cases} \boldsymbol{\epsilon}(\mathbf{x}) = \frac{1}{2}\left\{\nabla \otimes \mathbf{u}(\mathbf{x}) + (\nabla \otimes \mathbf{u}(\mathbf{x}))^t\right\} \\ \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{x}) = \mathbb{C}(\mathbf{x}) : \boldsymbol{\epsilon}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) = \mathbf{E}.\mathbf{x} + \mathbf{u}_{per}(\mathbf{x}) \end{cases} \tag{7}$$

where the displacement field $\mathbf{u}_{per}$ is Ω-periodic and $\mathbb{C}(\mathbf{x})$ is the elasticity tensor satisfying the periodicity condition:

$$\begin{cases} \mathbb{C}(\mathbf{x}) = \mathbb{C}(\mathbf{x} + \mathbf{d}) \\ \mathbf{d} = \sum_{i=1}^{3} 2n_i a_i \mathbf{e}_i \end{cases} \tag{8}$$

where n_i is an arbitrary integer. Strain and stress tensors are also periodic:

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x} + \mathbf{d}) \\ \boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}(\mathbf{x} + \mathbf{d}) = \mathbf{E} + \boldsymbol{\epsilon}_{per} \end{cases} \tag{9}$$

3.2. Strain and stress fields in Fourier space

Because of the periodicity of the medium, the solution can be developed into Fourier series, as proposed by Iwakuma and Nemat-Nasser (1983) or Moulinec and Suquet (1994).

Let us consider a periodic function $f(\mathbf{x})$ defined on the cell Ω defined by:

$$\Omega = \{\mathbf{x}, -a_j \leqslant x_j \leqslant a_j\ (j = 1,2,3)\} \tag{10}$$

with the condition of periodicity: $f(\mathbf{x}) = f(\mathbf{x} + \mathbf{d})$

This function can be expanded into Fourier series as follows:

$$f(\mathbf{x}) = \sum_{\xi} \hat{f}(\xi) e^{i\xi\mathbf{x}}, \quad i = \sqrt{-1} \tag{11}$$

with:

$$\hat{f}(\xi) = \frac{1}{V}\int_V f(\mathbf{x}) e^{-i\xi\mathbf{x}} dV, \quad \xi_j = \frac{\pi n_j}{a_j}, \quad (\text{no sum on } j) \tag{12}$$

Let us consider the periodic part $\mathbf{u}_{per}$ of the displacement field $\mathbf{u}$ those constant part is assumed null:

$$\mathbf{u}_{per}(\mathbf{x}) = \sum_{\xi}{}' \widehat{\mathbf{u}_{per}}(\xi) e^{i\xi\mathbf{x}} \tag{13}$$

where a prime on Σ indicates that $n = \sqrt{n_k n_k} = 0$ is excluded from the summation. Each Fourier component is given by:

$$\widehat{\mathbf{u}_{per}}(\xi) = \frac{1}{V}\int_V \mathbf{u}_{per}(\mathbf{x}) e^{-i\xi\mathbf{x}} d\mathbf{x} \tag{14}$$

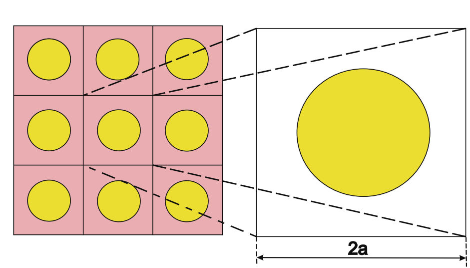

Fig. 1. Basic cell of a heterogeneous periodic medium.