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On the asymptotic crack-tip stress fields in nonlocal orthotropic elasticity

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ABSTRACT

The crack-tip stress fields in orthotropic bodies are derived within the framework of Eringen's nonlocal elasticity via the Green's function method. The modified Bessel function of second kind and order zero is considered as the nonlocal kernel. We demonstrate that if the localisation residuals are neglected, as originally proposed by Eringen, the asymptotic stress tensor and its normal derivative are continuous across the crack. We prove that the stresses attained at the crack tip are finite in nonlocal orthotropic continua for all the three fracture modes (I, II and III). The relative magnitudes of the stress components depend on the material orthotropy. Moreover, non-zero self-balanced tractions exist on the crack edges for both isotropic and orthotropic continua. The special case of a mode I Griffith crack in a nonlocal and orthotropic material is studied, with the inclusion of the T-stress term.

1. Introduction

1.1. Nonlocal elasticity

Nonlocal elasticity is based on an integral form of the stress–strain constitutive equations. In nonlocal elasticity, the stress at a point is expressed via a weighted average of the overall strain field (Polizzotto, 2001). Eringen (1972, 1974) postulated the constitutive equations of a nonlocal and linear elastic continuum in the following form:

$$\sigma_{ij}(\underline{x}) = \int_V \hat{\sigma}_{ij}(\underline{x}')\alpha(|\underline{x} - \underline{x}'|)dV, \tag{1}$$

where $\underline{x}$ and $\underline{x}'$ represent two positions within a body having volume V; σ_{ij} are the nonlocal stresses, while $\hat{\sigma}_{ij}$ represents the stress tensor for a classical (i.e. local) Hookean solid. In Eringen's integral constitutive Eq. (1), the weight function $\alpha(|\underline{x} - \underline{x}'|)$ is denoted as "kernel". Kernel functions have a dimension of length^{-3}, and therefore characteristic length scales can be directly associated with nonlocal continua. Kernels must satisfy some fundamental properties (Eringen, 1974); many alternative and legitimate choices for the selection of kernels do exist, each one leading to a different nonlocal theory. Nonetheless, nonlocal elasticity reverts to the classical theory when a Dirac's delta function is chosen as kernel.

Let us assume that the elastic fields are slowly varying with respect to the aforementioned length scales. Eringen (1974, 2002)) demonstrated that the integral operator that appears in Eq. (1) can be replaced by an equivalent differential operator. In this case, the nonlocal constitutive equations become:

$$\left(1 - \ell^2\nabla^2 + \gamma^4\nabla^4\right)\sigma_{ij} = \hat{\sigma}_{ij}, \tag{2}$$

where ℓ and γ represent nonlocal length scales, while $\hat{\sigma}_{ij}$ is again the Hookean stress tensor from classical (i.e. local) elasticity. The nonlocal length scales ℓ and γ can be identified by matching experimental dispersion curves for wave propagation in solids (Eringen, 1972).

As an alternative, let us consider the case in which $\hat{\sigma}_{ij} = \bar{\sigma}_{ij}$, where $\bar{\sigma}_{ij}$ is the stress tensor that satisfies the equilibrium equations of classical elasticity. In this case, the nonlocal stresses can be obtained via integrating the nonlocal differential constitutive equations (Eringen, 2002). We shall focus on this alternative throughout this paper. However, this alternative approach requires the introduction of appropriate boundary conditions for the nonlocal stresses.

Eringen also considered a simplified form of Eq. (2), obtained by setting $\gamma = 0$. This condition leads to the following set of nonlocal constitutive equations (Eringen, 2002, p. 100):

$$\left(1 - \ell^2\nabla^2\right)\sigma_{ij} = \bar{\sigma}_{ij}. \tag{3}$$

It is worth observing that the nonlocal stress σ_{ij} is the "true" stress that must be determined by satisfying both the equilibrium and the constitutive equations that govern nonlocal elasticity. Clearly, the classical $\underline{\bar{\sigma}}$ and the nonlocal stresses $\underline{\sigma}$ are respectively associated with two different continuum models, so they cannot coexist within the same elastic body. However, since both classical and nonlocal stresses appear in the constitutive equations Eq. (3), we shall

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remark the distinction between the former and the latter throughout the paper. It has to be kept in mind that the classical stress is given, while the nonlocal field has to be determined.

The kernel associated with the constitutive equations Eq. (3) for plane and anti-plane problems is (Eringen, 2002):

$$\alpha(|\underline{x}-\underline{x}'|)=\frac{1}{2\pi\ell^2}K_0\left(\frac{|\underline{x}-\underline{x}'|}{\ell}\right), \quad (4)$$

where K_0 is a modified Bessel function of the second kind and order zero. Eringen also demonstrated that the kernel in Eq. (4) is the Green's function of the differential operator $\left(1-\ell^2\nabla^2\right)$ (Eringen, 2002, p. 105). Eq. (3) was applied to derive the nonlocal stress field for a mode I Griffith crack (Ari and Eringen, 1983; Eringen et al., 1977) and for describing the interaction between a dislocation and a crack in anti-plane shear (Eringen, 1983). The procedure followed by Eringen was based on substituting the local stress field from linear elastic fracture mechanics (LEFM) into the right hand side of Eq. (3). He assumed that, when dealing with crack problems, the same boundary conditions were applicable to both local and nonlocal elasticity. Wang (1992) proposed an approximated method for computing the crack-tip stresses that simplified the calculations based on Eringen's theory. Yang (2009) extended Eringen's solution to a mode I Griffith crack in an orthotropic material. Eringen also considered the case of a line crack subjected respectively to plane shear (Eringen, 1978) and anti-plane shear (Eringen, 1979). However, in this cases he employed a different kernel from the one shown in Eq. (4). Zhou et al. (1999) addressed also the nonlocal anti-plane shear problem proposing a solution scheme based on Schmidt's method.

The key result obtained from nonlocal elasticity is that the stress field at the crack tip is finite. This is true for all the cases listed above. Moreover, the nonlocal stresses attain a finite maximum value at a characteristic distance ahead of the crack tip. Eringen (2002, p. 136) estimated such distance to be 0.5ℓ for a mode I Griffith crack in an isotropic material. The stress intensity factors (SIFs) defined in linear elastic fracture mechanics (LEFM) correspond to finite stress concentrations in nonlocal elasticity. Therefore, fracture can be predicted employing only stress-based criteria (Eringen, 2002, pp. 137–141).

1.2. Stress-gradient versus strain-gradient theories

A different class of nonlocal theories arise when the stress is considered to be a function of the strain gradients at a point. These gradient elasticity models are considered to be "weakly nonlocal" (Polizzotto, 2001) in relation to the "strongly nonlocal" theories based on the integral constitutive Eq. (1). Strain gradient elasticity originated from the Cosserats' "first-gradient" micro-polar continuum model (Cosserat and Cosserat, 1909; Fannjiang et al., 2002). Second-order strain gradient theories were introduced by Mindlin (1965) and Casal (1972), in order to represent surface effects on the potential energy of the elastic continuum. Casal's approach (Casal, 1972) is based on considering two characteristic length scales in the strain gradient constitutive equations, which describe size and scale effects associated with material micro-structures (Paulino et al., 2003). In weakly nonlocal elasticity the strain singularity at crack tips is suppressed (Altan and Aifantis, 1992; Ru and Aifantis, 1993), but the stress singularity is retained. Both Zhang et al. (1998) and Fannjiang et al. (2002) found stress singularities of order $r^{-3/2}$ for mode III cracks in second order strain-gradient elasticity. A similar behaviour of the stress singularity was also observed for mode III cracks in functionally graded materials (Paulino et al., 2003; Chan et al., 2008). It is also possible to envisage more general nonlocal theories (Aifantis, 2003), where the integral nonlocal stresses in Eq. (1) depend on the strain tensor and its gradients, thus coupling "weakly" and "strongly" nonlocal continuum mechanics. In principle, this may lead to the suppression of both stress and strain singularities at crack tips.

1.3. The boundary conditions conundrum

All the results discussed in Section 1.2 were obtained assuming that the LEFM boundary conditions for the crack problem were directly applicable within the framework of "strongly" nonlocal elasticity. In Refs. Ari and Eringen (1983), Eringen et al. (1977), Eringen (1978, 1979, 2002), Wang (1992) and Zhou et al. (1999), the crack edges were loaded with constant tractions and the far-field stresses were set to zero. This is equivalent to consider zero edge tractions with far-field stresses opposite to those acting on the loaded cracks. We will refer to the latter case throughout this paper, without any loss of generality.

Atkinson (1980a,b) criticised Eringen's approach to study cracks by means of nonlocal elasticity. He demonstrated that Eringen's results were based on non-uniform approximations due to the LEFM boundary conditions adopted. He pointed out that the solution of the nonlocal crack problem as expressed by Eringen may not exist because nonlocal stresses are influenced by the diffusive nature of the Laplacian operator that appears in the constitutive relations in Eq. (3). The latter require that the stress field be C^2 in the solution domain for a non-singular solution to exist. Therefore, in nonlocal elasticity, it is not possible to postulate that the stress field jumps from zero to some finite value just across the crack tip. On the contrary, if finite stresses exist at the crack tip, then tractions must be present on the crack edges. These tractions are due to "finite-range" elastic interactions, whose characteristic length scale is the nonlocal distance ℓ that appears in Eqs. (3) and (4).

Cheng (1991) assumed that the crack edge tractions were zero only in the limit $\ell \to 0$. This was done to overcome the difficulties associated with the boundary conditions posited in Eringen's approach to crack-tip problems (Ari and Eringen, 1983; Eringen et al., 1977; Eringen, 1978, 1979; Wang, 1992; Zhou et al., 1999). However, Cheng's results also gave a nonlocal stress field with non-zero tractions on the crack edges.

In order to provide a visual insight about the issues discussed above, let us consider a crack whose tip is located at $x = 0$ (Fig. 1). The qualitative trends of the crack-tip stress obtained from LEFM and nonlocal elasticity are presented in the same figure. Let us assume that the stress component σ is associated with the traction vector on the crack edges, independently from the actual crack

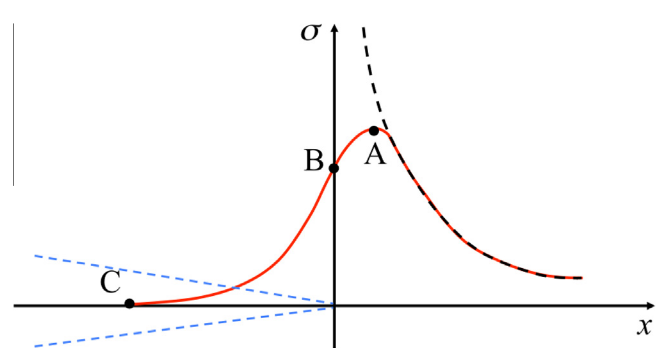

Fig. 1. Qualitative distribution of crack-tip stresses from Ref. Eringen (2002), Fannjiang et al. (2002), Gao and Chiu (1992), Kim and Paulino (2004), Mindlin (1965), Paulino et al. (2003), Polizzotto (2001), Prabhu and Lambros (2002), Ru and Aifantis (1993), Sih et al. (1965), Sun and Jin (2012), Suo (1990), Wang (1992), Williams (1952), Yang (2009), Zhang et al. (1998) and Zhou et al. (1999). LEFM (black dashed line) and nonlocal elasticity (continuous red line) trends; crack tip position at $x = 0$, i.e. point B. A is the maximum stress point. C is the cut-off point for the traction on the crack edges. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)