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Superposing scheme for the three-dimensional Green's functions of an anisotropic half-space

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ABSTRACT

This paper presents a method of superposition for the half-space Green's functions of a generally anisotropic material subjected to an interior point loading. The mathematical concept is based on the addition of a complementary term to the Green's function in an anisotropic infinite domain. With the two-dimensional Fourier transformation, the complementary term is derived by solving the generalized Stroh eigenrelation and satisfying the boundary conditions on the free surface with the use of Green's functions in the full-space case. The inverse Fourier transform leads to the contour integrals, which can be evaluated with the application of Cauchy residue theorem. Application of the present results is made to obtain analytical expression for the orthotropic materials which were not reported previously. The closed-form solutions for the transversely isotropic and isotropic materials derived directly from the solutions as being a special case are also given in this paper.

1. Introduction

The elastic Green's function for an anisotropic material provides the fundamental solution for the anisotropic elastic theory. Green's functions represent a key quantity for the modeling of material properties and the analysis of related elastic fields in inclusion and dislocation problems of anisotropic materials. It is well known that Green's functions can be obtained explicitly only for isotropic and transversely isotropic materials. The three-dimensional Green's function for an anisotropic material is much more complicated to obtain than the isotropic one; moreover, the half-space Green's functions impose further difficulties on fulfilling the conditions of free surface. Much effort has been devoted to deriving explicit expressions, approximate forms, or numerical evaluations of Green's functions and their derivatives.

For the half space Green's functions, the analyses are usually based on either potential functions method or Fourier transform method. By using the method of potential functions, the elastic fields are expressed in terms of potential functions so that the equilibrium equations are satisfied. The case due to a concentrated force applied at an interior point of an isotropic half space has been obtained by Mindlin (1936, 1953) using the Galerkin functions and by Rongved (1955) using the Papkovich–Neuber displacements. Pan and Chou (1979) gave explicit expressions of the half space Green's function for a transversely isotropic material. As in the isotropic case, the solutions are obtained by superposing certain harmonic and bi-harmonic functions so that all boundary condition are satisfied. Lin et al. (1991) presents the response of a transversely isotropic half space subjected to various distributions of normal and tangential contact stresses on its surface also by using the method of potential functions. The Fourier transform method was used to solve mixed boundary value problem of Boussinesq type for a generally anisotropic half space by Willis (1966, 1967) and for a hexagonally aeolotropic elastic half space by Sneddon (1992). The Green's function of a point force applied at the surface of a semi-infinite generally anisotropic solid has been developed by Barnett and Lothe (1975) using the Stroh formalism and Fourier transform technique. Dealing with the half-space Green's function of a hexagonal continuum, Lee (1979) brought up with an idea by decomposing it into the Green's function in an infinite body and a supplementary form to satisfy boundary conditions on the free surface. The same technique was later used by Pan (2003) to examine the Green's function in an anisotropic half space with various boundary conditions.

Following the superposing method by Lee (1979), the Green's function due to a point force inside a generally anisotropic half space is evaluated in this paper. Among them the explicit expressions of three-dimensional Green's functions of a point force in an infinite generally anisotropic solid have been derived by Ting and Lee (1997). The central problem for explicitly solving the Green's function depends upon the roots, which is called the Stroh eigenvalues, of a characteristic sextic algebraic equation. By using two-dimensional Fourier transforms method and satisfying the

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boundary conditions on the free surface, the supplementary term is obtained in the form of generalized Stroh formalism also by making use of Green's functions for the infinite solids. Section 2 gives the concept of superposition method. The review of Green functions for infinite domain of a general anisotropic material is given in Section 3. The generalized Stroh eigenrelation formulation and inverse algorithm for determination of the complementary terms are presented in Section 4. For illustration purpose, the method was used to obtain explicit expressions for the half space Green's functions for transversely isotropic materials in Section 5. A special case is included when the loading is applied on the free surface.

2. Basic formulation

Let a semi-infinite domain is defined so that the half space occupies $x_3 \geqslant 0$ and the origin of coordinates is taken at the boundary of the half space. Consider the half space subjected to a concentrated force in an interior point $(0,0,h)$, the three-dimensional elastic Green's function formulation for a generally anisotropic material will be discussed. The Green's function should satisfy the equations of equilibrium and traction free conditions on the free surface as

$$C_{ijkl}G_{ks,jl} + \delta_{is}\delta(x_1)\delta(x_2)\delta(x_3 - h) = 0, \quad x_3 \geqslant 0, \tag{2.1}$$

$$C_{i3kl}G_{ks,l} = 0, \quad (i,j,k,l = 1,2,3) \quad x_3 = 0, \tag{2.2}$$

where C_{ijkl} are the elastic moduli, $\delta(\,)$ is the Dirac delta function, δ_{is} is the Kronecker delta, and a comma denotes differentiation. Lee (1979) proposed that the Green's function $\boldsymbol{G}$ for a half space can be decomposed into two parts as

$$\boldsymbol{G} = \boldsymbol{G}^{\infty} + \boldsymbol{G}^{c}, \tag{2.3}$$

where $\boldsymbol{G}^{\infty}$ is the Green's function for the infinite space with a point load applied at (0,0,h), and the complementary part $\boldsymbol{G}^{c}$ should satisfy

$$C_{ijkl}G^{c}_{ks,jl} = 0, \quad x_3 \geqslant 0, \tag{2.4}$$

$$C_{i3kl}G^{c}_{ks,l} = -C_{i3kl}G^{\infty}_{ks,l}, \quad x_3 = 0, \tag{2.5}$$

3. The solutions of $\boldsymbol{G}^{\infty}$

We begin by introducing the three-dimensional elastic Green's function $\boldsymbol{G}^{\infty}$ due to a point load applied interior to a generally anisotropic infinite medium. The detail discussion of the explicit solutions can be found in Ting and Lee (1997) and Lee (2002). Here we give a brief review of the fundamental formulation for the Green's function $\boldsymbol{G}^{\infty}$. Assuming that a point load is applied at an interior point $(0,0,h)$ of an infinite domain, the equation of equilibrium can be written in the absence of body force as

$$C_{ijkl}G^{\infty}_{ks,jl} + \delta_{is}\delta(x_1)\delta(x_2)\delta(x_3 - h) = 0, \quad x_3 \geqslant 0. \tag{3.1}$$

The Fourier transformations of $\boldsymbol{G}^{\infty}$ with respect to x_1, x_2 and x_3 are defined by

$$\widetilde{\boldsymbol{G}}^{\infty}(\boldsymbol{y}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{G}^{\infty}(\boldsymbol{x})e^{i\mathbf{y}\cdot\mathbf{x}}dx_1\,dx_2\,dx_3, \tag{3.2}$$

where $\boldsymbol{x} = (x_1,x_2,x_3)$ is the position vector, and $\boldsymbol{y} = (y_1,y_2,y_3)$ is the transformed parameters. On taking Fourier transforms on (3.1), we have

$$L_{ik}\widetilde{G}^{\infty}_{ks}(y_1,y_2,y_3) = (2\pi)^{-3/2}\delta_{is}e^{iy_3h}, \tag{3.3}$$

with the component $L_{ik} = C_{ijkl}y_jy_l$. In spherical coordinate system, we may choose

$$x_1 = r\sin\phi\cos\theta, \quad x_2 = r\sin\phi\sin\theta, \quad x_3 - h = r\cos\phi, \tag{3.4}$$

where the vector $\boldsymbol{r} = (x_1,x_2,x_3 - h)$ and $r = |\boldsymbol{r}|$. In this notation, the θ is measured on the plane $x_3 = h$ and $\phi = 0$ is directing to the positive x_3 direction. Let $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ be any two mutually orthogonal unit vectors on the oblique plane whose normal is the vector $\boldsymbol{r}$, and we choose $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ as

$$\boldsymbol{n}^* = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi), \quad \boldsymbol{m}^* = (-\sin\theta, \cos\theta, 0). \tag{3.5}$$

The vectors $[\boldsymbol{n}^*, \boldsymbol{m}^*, \boldsymbol{r}]$ form a right-handed triad. Then, the application of inverse transforms on (3.3) leads to (Synge, 1957; Barnett, 1972; Ting and Lee, 1997)

$$\boldsymbol{G}^{\infty}(\boldsymbol{x}) = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\boldsymbol{\Gamma}^{-1}(p)dp = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{\hat{\boldsymbol{\Gamma}}(p)}{|\boldsymbol{\Gamma}(p)|}dp, \tag{3.6}$$

where

$$\boldsymbol{\Gamma}(p) = \mathbf{Q}^* + p(\mathbf{R}^* + \mathbf{R}^{*T}) + p^2\mathbf{T}^*, \tag{3.7}$$

$$Q^*_{ik} = C_{ijkl}n^*_j n^*_l, \quad R^*_{ik} = C_{ijkl}n^*_j m^*_l, \quad T^*_{ik} = C_{ijkl}m^*_j m^*_l \tag{3.8}$$

and $\hat{\boldsymbol{\Gamma}}$ represents the adjoint matrix of $\boldsymbol{\Gamma}\ (p)$. The determinant of $\boldsymbol{\Gamma}(p)$ can be written explicitly as $|\boldsymbol{\Gamma}(p)| = |\mathbf{T}^*|f(p)$ and the vanishing of the determinant $|\boldsymbol{\Gamma}(p)| = 0$ leads to a sextic equation in p, which has six independent roots in general. With the condition of positive definite $\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0$, it can be shown that the roots are complex and are three pairs of complex conjugates as

$$f(p) = (p - p_1)(p - p_2)(p - p_3)(p - \bar{p}_1)(p - \bar{p}_2)(p - \bar{p}_3). \tag{3.9}$$

Therefore, Eq. (3.9) can be decomposed in terms of the six roots as with $p_v = \alpha_v + i\beta_v\ (v = 1,2,3)$ denotes the complex roots with a positive imaginary part, and the overbar is its complex conjugate. For a general anisotropic material the Stroh eigenvalues p_v of the sextic Eq. (3.9) depends on the material constants and position vector parameters. Ting and Lee (1997) examined the integral formula in (3.6) and presented the explicit expressions of the Green's function for a general anisotropic material. The analytical solution of (3.6) depends on the roots of the sextic algebraic equation. With Cauchy's theory of residues, the Green's function can be expressed as

$$G^{\infty}_{ij} = \frac{i}{2\pi r|\mathbf{T}^*|}\sum_{v=1}^{3}\frac{\hat{\Gamma}_{ij}(p_v)}{f'(p_v)}, \tag{3.10}$$

where $f'(p_v) = \{df(p)/dp\}_{p=p_v}$. The adjoint matrix $\hat{\boldsymbol{\Gamma}}(p)$ is a polynomial in p of degree four. Let

$$\hat{\boldsymbol{\Gamma}}(p) = \sum_{n=0}^{4}p^n\hat{\boldsymbol{\Gamma}}^{(n)}, \tag{3.11}$$

where the real matrices $\hat{\boldsymbol{\Gamma}}^{(n)}$ are independent of p. With the expansion of $\hat{\boldsymbol{\Gamma}}(p)$, the Green's function can then be written as (Ting and Lee, 1997)

$$\boldsymbol{G}^{\infty}(x) = \frac{1}{4\pi r|\mathbf{T}^*|}\sum_{n=0}^{4}q_n\hat{\boldsymbol{\Gamma}}^{(n)} = \frac{1}{4\pi r}\mathbf{H}[\theta,\phi], \tag{3.12}$$

where q_n is shown in Appendix A. This is the explicit Green's function expression in terms of the Stroh eigenvalues for any given anisotropic material. Here we can see that the key step for solving the three-dimensional elastic Green's function of an anisotropic material depends upon the roots $p_v\ (v = 1,2,3)$ of a sextic equation in (3.9). For generalized anisotropic materials, a sextic equation cannot be solved analytically and only numerical solutions are possible. However, in materials with higher symmetry or with special relations existing among material constants, the difficulty is reduced and analytical solution become possible. The most valuable