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Circular loads on the surface of a half-space: Displacement and stress discontinuities under the load

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ABSTRACT

Based on the expressions for the surface displacements due to concentrated vertical and tangential forces acting on the free surface of a half-space, available from the well-known Boussinesq and Cerruti elasticity problems, the surface displacements and the surface stresses are derived for a half-space loaded by the vertical and tangential circular ring loads, or by uniform normal and radial shear stresses applied within a circular or annular circular domains. By using different routes of integration, alternative forms of displacement expressions are derived from the concentrated force results. Betti's reciprocal theorem is used to relate the displacements due to radial and vertical ring loads. The displacement and stress discontinuities under these loads, or along the boundaries of the circular domains within which the uniform stress is applied, are evaluated and discussed. The radial and circumferential components of stress are discontinuous under the load whenever the slope of the radial displacement is discontinuous under that load.

1. Introduction

The present work was motivated by recent studies devoted to the determination of the deformed shape of the surface of a soft substrate due to deposited liquid drop (Pericet-Camára et al., 2008; Yu and Zhao, 2009; Liu et al., 2009; Roman and Bico, 2010; Olives, 2010; Das et al., 2011; Jerison et al., 2011; Lubarda and Talke, 2011; Lubarda, 2012). If the solid substrate is sufficiently soft, the distributed capillary force along the triple contact line between the solid/liquid/vapor phase, resulting from the surface tensions and intermolecular interactions around the triple contact (Fig. 1), can give rise to appreciable uplifting of the surface of the substrate below the triple contact line. The formation of circular ridges can have significant effects on the functioning of MEMS and other micro/nano devices, lubrication of magnetic hard disks, molten solder spreading in electronic packaging, etc. (Carré et al., 1996). This was studied by using a linear elasticity theory by many researchers, with the early contributions by Lester (1961) and Rusanov (1975), followed by Fortes (1984), Shanahan (1988) and Kern and Muller (1992). The elastic response in this problem is characterized by the singularity of the vertical component of displacement below the capillary force, assumed to be distributed as a circular line load. The elasticity solution also predicts a discontinuous radial displacement under the vertical line load. To eliminate this singularity, one approach is to distribute the capillary force within a finite width, related to the actual thickness of the interface liquid/vapor layer and the molecular interactions between a liquid drop and a solid substrate. This thickness may vary from 1 nm for harder substrates to microns for softer rubber or gel substrates (Lester, 1961; Rusanov, 1975; de Gennes, 1985; Yu and Zhao, 2009), but is, in any case, much smaller than the radius R of the contact circle (Fig. 1). Even though such procedure eliminates the vertical displacement singularity, it does not eliminate the discontinuity in the slope of the radial displacement at the boundaries of the annular circular region within which the capillary force is distributed, and this gives rise to the discontinuity in both the radial and circumferential stresses across these boundaries. The effect of stress on the wetting angle was studied by Srolovitz and Davis (2001), who found that elastic effects in solids are incapable of modifying the wetting angle determined by interfacial tensions, except in crack-like geometries. Recently, Style and Dufresne (2012) examined the effect of the surface tension and the elastocapillary length on the peak displacement under the load. Marchand et al. (2012) determined the effective surface tension from the elastic displacement field of a thin elastomeric wire immersed in a liquid bath, observing experimentally an unexpected direction of the force transmission along the contact line.

The stress discontinuity also arises in the classical Love's (1929) problem of a semi-infinite solid loaded by a uniform pressure p within a circular area, in which the radial and circumferential stress discontinuities across the loading boundary are of magnitude p and

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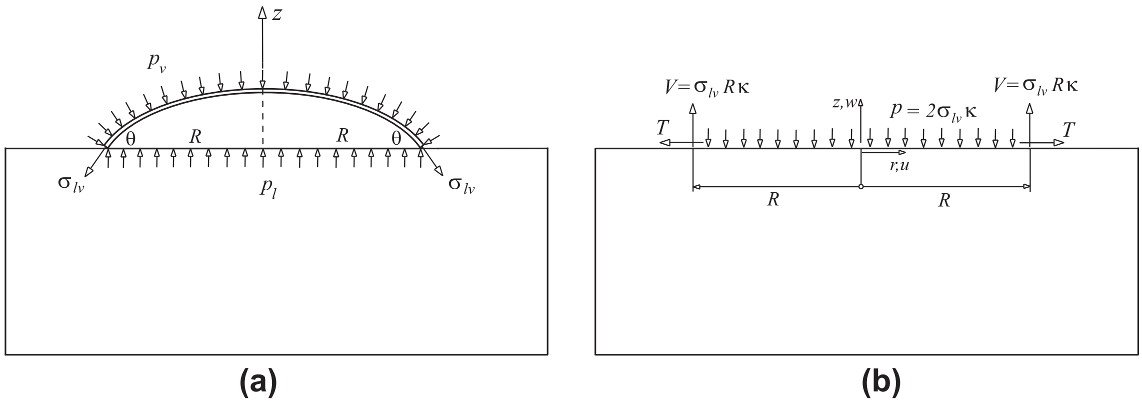

Fig. 1. (a) A free-body diagram of a liquid drop. The liquid pressure is p_l, the vapor pressure is p_v, and the liquid/vapor surface tension is σ_{lv}. (b) A self-equilibrated loading on the surface of the substrate, consisting of pressure $p = 2\kappa\sigma_{lv}$, vertical line force $V = \sigma_{lv}\sin\theta = R\kappa\sigma_{lv}$, and the tangential (radial) line force $T = \sigma_{lv}\cos\theta$, where θ is the Young's contact angle. The liquid/vapor surface tension is σ_{lv}. The mean curvature of the drop around the triple contact line of radius R is κ.

$2\nu p$, respectively. See also Sneddon (1951) discussion of Terezawa (1916) solution. In the case of uniform radial shear stress applied within the circle of radius R, the slope of the radial displacement becomes infinite at the center ($r = 0$) and along the boundary $r = R$, which results in the stress singularities at these points as well. If the shear stress is distributed within an annular region $R_0 \leqslant r \leqslant R$, the singularity at the center is eliminated, but the stresses are still singular along the circles $r = R_0$ and $r = R$. In the limit as $R_0 \to R$, the solution for the radial ring load is recovered, for which the radial displacement and its slope, and thus the radial and circumferential stresses, are all singular under the load, while the vertical displacement is finite but discontinuous.

The solutions for some of the elasticity problems considered in this paper have been previously constructed and reported in the literature, e.g., Sneddon (1951), Timoshenko and Goodier (1970) and Johnson (1985), or can be deduced from them by an appropriate integration, but the stress and displacement discontinuities, inherently imbedded in these solutions, were not fully discussed or examined. Furthermore, the expressions for the surface displacement and stress components for all loadings considered in this paper are derived by using the results for the surface displacements due to the concentrated vertical (Boussinesq) or tangential (Cerruti) force only, without resorting to involved solutions of the corresponding entire boundary value problems. Different expressions for the displacement components due to vertical and tangential ring loads are derived and discussed.

2. Surface displacement components due to concentrated force

For the later use in the paper, we list in this section the expressions for the surface displacements in the well-known Boussinesq and Cerruti concentrated force problems, and the surface displacements from the surface doublet and quadruplet acting on the boundary of an elastic half-space.

2.1. Boussinesq problem

In the Boussinesq elasticity problem, the displacement components of the points of the bounding surface ($z = 0$) of a half-space, due to the applied concentrated vertical force Q_z are given by (e.g., Johnson, 1985, p. 50)

$$u_\xi = \frac{Q_z(1-2\nu)}{4\pi G}\frac{\xi}{\rho^2}, \quad u_\eta = \frac{Q_z(1-2\nu)}{4\pi G}\frac{\eta}{\rho^2}, \quad u_z = \frac{Q_z(1-\nu)}{2\pi G}\frac{1}{\rho}, \tag{1}$$

where G is the elastic shear modulus and ν is the Poisson ratio. The in-plane Cartesian coordinates are (ξ, η) and ρ is the radial distance from the origin at which the force is applied. The radial displacement is accordingly

$$u_\rho = \frac{Q_z(1-2\nu)}{4\pi G}\frac{1}{\rho}. \tag{2}$$

The nonvanishing surface strain components are

$$\epsilon_\rho = \frac{du_\rho}{d\rho} = -\frac{Q_z(1-\nu)}{2\pi G}\frac{1}{\rho^2}, \quad \epsilon_\theta = \frac{u_\rho}{\rho} = \frac{Q_z(1-\nu)}{2\pi G}\frac{1}{\rho^2} \equiv -\epsilon_\rho. \tag{3}$$

Since $\sigma_z = 0$ away from the load, the corresponding surface stress components are, from Hooke's law, $\sigma_\theta = -\sigma_\rho = 2G\epsilon_\theta$.

2.2. Cerruti problem

The displacement components due to the concentrated tangential force Q_ξ are (Cerruti problem; Johnson, 1985, p. 69)

$$u_\xi = \frac{Q_\xi}{2\pi G}\left(\frac{1}{\rho} - \nu\frac{\eta^2}{\rho^3}\right), \quad u_\eta = \frac{Q_\xi \nu}{2\pi G}\frac{\xi\eta}{\rho^3}, \quad u_z = -\frac{Q_\xi(1-2\nu)}{4\pi G}\frac{\xi}{\rho^2}. \tag{4}$$

The vertical displacement along the ξ-axis is singular and discontinuous at $\xi = 0$. The points $\xi > 0$ are depressed ($u_z < 0$), while the points $\xi < 0$ are elevated ($u_z > 0$). Note that $u_\xi^{-Q_z} = u_z^{Q_\xi}$, provided that the magnitude of the compressive force $(-Q_z)$ is equal to the magnitude of the shear Q_ξ; cf. (1) and (4). It is recalled that in a two-dimensional (plain-strain) version of the problem, the vertical displacement due to tangential concentrated force is finite but discontinuous under the force, $u_z = -Q_x(1-2\nu)\mathrm{sgn}(x)/(4G)$. Likewise, in a two-dimensional Flamant's problem, the horizontal displacement due to concentrated vertical force is $u_x = Q_z(1-2\nu)$ $\mathrm{sgn}(x)/(4G)$.

2.3. Surface doublet

Two co-linear tangential forces Q at small distance d constitute a doublet of forces shown in Fig. 2(a). By superposition of results from (4), the displacement components are

$$u_\xi = \frac{Q}{2\pi G}\left(\frac{1}{\rho_1} - \nu\frac{\eta^2}{\rho_1^3} - \frac{1}{\rho_2} + \nu\frac{\eta^2}{\rho_2^3}\right),$$
$$u_\eta = \frac{Q\nu\eta}{2\pi G}\left(\frac{\xi - d/2}{\rho_1^3} - \frac{\xi + d/2}{\rho_2^3}\right), \tag{5}$$