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Polar plasticity of thin-walled composites made of ideal fibre-reinforced material

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ABSTRACT

The present study investigates the influence that polar material response has on the plastic behaviour of thin-walled structures made of ideal fibre-reinforced materials (Spencer, 1972); or, equivalently, on the response of thin-walled fibrous composites within the first branch of the matrix dominated form (MDM) of the bimodal theory of plasticity (Soldatos, 2011; Dvorak and Bahei-El-Din, 1987). The plasticity studies mentioned above assume that fibres are infinitely thin and, therefore, perfectly flexible. They possess no bending stiffness and, hence, their negligible bending resistance cannot influence the developed stress state, which is accordingly described by a symmetric stress tensor. In contrast, the present study considers that if fibres resistant in bending are embedded in a material at high volume concentrations, their flexure produces couple-stress and, as a result of this kind of polar material behaviour, the stress tensor becomes non-symmetric. Under plane stress conditions that dominate behaviour of thin-walled structures, the stress-space and, therefore, conditions of plastic yield and relevant yield surfaces are thus four-dimensional. However, shapes and properties of initial yield surfaces relevant to the f_1-branch of MDM are studied comprehensively by considering their projection on particular planes of such a four-dimensional stress-space. It then becomes easier understood that, in the regime of polar material response, a thin-walled structure made of ideal fibre-reinforced material deforms plastically when suitable combinations of shear stress values are reached simultaneously, rather than when only one of two unequal shear stress components reaches some maximum absolute value. Thus, polar material plasticity dismisses the conventional concept of material yield stress in shear and replaces it with a pair of two independent yield moduli. Existence of the latter is perceived as a theoretical justification of the expectation that, due to the presence of fibres, two rather than one shear yield parameters of the composite should be present and accountable for. The non-zero values of those parameters are shown to exert paramount influence on the form of the yield surface of the ideal fibre-reinforced material of interest.

1. Introduction

The plasticity theory of ideal fibre-reinforce materials was initiated by Spencer and his collaborators (Mulhern et al., 1967) and communicated for publication by R. Hill, FRS, at a time that the set of corresponding experimental data available was essentially negligible. Despite the good and encouraging agreement of its predictions with that particularly limited set of available experimental data (Cooper, 1966; Jackson and Cratchley, 1966), Spencer (1992) admittedly quoted later that “the theory had to be regarded in a rather tentative manner” for approximately two decades; that is, until the bimodal theory of plasticity became available (Dvorak and Bahei-El-Din, 1987) and its validity was underpinned by a comprehensive amount of relevant experimental measurements (see also Dvorak et al., 1988, 1991; Nigam et al., 1994a, 1994b).

The striking similarity between important characteristics of those two differently established plasticity models was noted immediately by Dvorak and Bahei-El-Din (1987), Dvorak et al. (1988), Dvorak (2000) and was also welcomed by Spencer (1992). However, it is only recently, almost two more decades later (Soldatos, 2011), that complete elucidation is achieved of the manner in which the experimentally established bimodal theory (Dvorak and Bahei-El-Din, 1987; Dvorak et al., 1988, 1991; Nigam et al., 1994a,b) relates physically and mathematically to the plasticity theory of ideal fibre-reinforced solids (Mulhern et al. 1967, 1969; Spencer, 1972, 1992).

Arguments of the invariant theory of tensor representations made accordingly clear (Soldatos, 2011) that activation of the matrix dominated mode (MDM) in bimodal plasticity is possible only if the applied stress state allows fibres to respond like they are practically inextensible. Moreover, activation of the more dominant, among the two MDM plastic slip branches is possible only if conditions of material incompressibility also apply, together with the implied constraint of fibre inextensibility. It is thus shown in

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(Soldatos, 2011) that within the regime of the more dominant branch of MDM, which is termed as the f_1-branch of MDM, the response of the fibrous composite is that of an ideal fibre-reinforced material (Spencer, 1972).

It is now emphasised that all of the plasticity studies mentioned above assume that fibres are infinitely thin and, therefore, perfectly flexible. They possess no bending stiffness and, hence, their negligible bending resistance cannot possibly influence the developed stress state; the later is accordingly described by a symmetric stress tensor. However, arguments associated with elastic deformation of ideal fibre-reinforced materials made recently evident (Soldatos, 2009b, 2010a; Dagher and Soldatos, 2011) that there are cases in which strong/stiff fibres may exhibit considerable bending resistance and, therefore, possess considerable bending stiffness. These results favour the postulates of a new hyperelasticity theory which accounts for fibre resistance in bending (Spencer and Soldatos, 2007). In particular, the fact that fibre bending resistance produces couple-stress and makes thus the stress tensor non-symmetric becomes a feature of predominant importance and influence in this investigation.

The anti-symmetric part of possible non-symmetric stress may be of considerable influence if fibres resistant in bending are embedded in a material at high volume concentrations and, therefore, cannot be always ignored or neglected. Nevertheless, such anti-symmetric stress contribution exerts direct influence on the shear stress components only which, in the case of plastic response, are of predominant importance in the description of the f_1-branch of MDM. Because the presence of couple-stress renders the fibre-reinforced composite of interest with properties of a polar material, it is appropriate to distinguish the present analysis from the afore mentioned “symmetric” plasticity studies, by referring to it as “non-symmetric” or “polar” plasticity of fibre-reinforced materials; for a definition of “polar” media see, for instance, (Truesdell and Noll, 1965; p. 389).

Due to stress non-symmetry ($\sigma_{ij} \neq \sigma_{ji}$), yield surfaces in three-dimensional polar plasticity are generally described in a nine- rather than a six-dimensional stress space; in the plane stress case, which is associated with plastic failure of thin-walled composites, these are described in a four- rather than a three-dimensional stress space. It follows that the symmetric plasticity counterpart of the actual yield surface of a polar material is only the projection of the actual yield surface in the corresponding symmetric stress space. This reduction of the actual dimensions of the stress space may result in the loss of important information regarding the plastic behaviour of the polar material of interest.

For instance, the conventional (symmetric plasticity) f_1-branch of MDM suggests that, when fibres are straight and aligned with the x_1-axis of a suitably chosen two-dimensional Cartesian co-ordinate system, the thin-walled fibrous composite of interest yields plastically as soon as the absolute value of the symmetric shear stress component ($\sigma_{21} = \sigma_{12}$) reaches some maximum (Spencer, 1972; Dvorak and Bahei-El-Din, 1987; Soldatos, 2011). However, when fibres are not perfectly flexible, the outlined polar plasticity arguments reveal that the ultimate, polar plasticity version of the MDM yield surface will be described in a four- rather than a three-dimensional stress space. Hence, plastic failure of the composite within its f_1-branch of MDM should rather be reached by suitable combinations of the values of σ_{21} and σ_{12} which, more generally, are expected to be unequal.

Under these considerations, the present study investigates the influence that the outlined polar plasticity concepts have on the response of thin-walled structures made of ideal fibre-reinforced material; or, equivalently, on the response of thin-walled fibrous composites within the f_1-branch of MDM of bimodal plasticity. In this context, Section 2 introduces basic theoretical concepts that underpin relevant developments under the plane stress conditions that dominate response of thin-walled structures. Section 2 serves also as a link between the outlined bimodal plasticity concepts for polar fibre-reinforced materials and relevant theoretical results stemming from the invariant theory of tensor representations. Those results are initially provided for fibres of general shape but they are also specialised for the particular case of straight fibres considered in experiments (Dvorak and Bahei-El-Din, 1987).

Postulates which are commonly employed in conventional plasticity theory favour yield surfaces which are (i) polynomials in the independent stress invariants, and (ii) quadratic in the stresses. Being employed in Section 3, these equip the present study with ability to search for and to establish the most general relevant form of a yield condition within the f_1-branch of bimodal polar material plasticity. The particular case of straight fibres makes thus understood that the projection (cross-section) of that yield surface on the aforementioned principal plane of stress-space, that is the $\sigma_{21}\sigma_{12}$-plane, will have the form of a conic section. Hence, the polar material version sought of the f_1-branch of MDM emerges as a choice among four potential yield surfaces; a pair of single-parameter and a pair of two-parameter surfaces.

The case of straight fibres becomes thus the pilot case of study in Section 4 and Section 5 where, for simplicity, one of the single- and one of the double-parameter surfaces, respectively, is considered and investigated thoroughly. Completeness of the present study requires a similarly comprehensive investigation of the remaining couple of a potential yield surfaces. However, for simplicity, this is outlined in corresponding Appendices because it resembles the analysis outlined in Section 4 and Section 5 to a considerable extent. A considerable and complete amount of valuable information is thus obtained, regarding the manner in which plastic slip is initiated in ideal fibre-reinforced composites that exhibit polar material behaviour.

Section 6 returns afterwards to and discusses the relevant form-invariant generalisation of the f_1-branch of the MDM, where fibres resistant in bending are considered to be of general shape; certain important features of an associated flow rule are also discussed in that section. Finally, Section 7 gives a comprehensive summary of the most important findings of this investigation.

2. Preliminary physical concepts and relevant mathematical foundation

Consider a thin layer of a unidirectional fibrous composite which is in a macroscopically uniform state of plane stress. The middle-plane of the layer coincides with the x_1x_2-plane of a Cartesian co-ordinate system Ox_i; the Ox_3-axis is directed upwards (Latin indices take the values 1, 2 and 3). The unidirectional family of fibres embedded in the layer is assumed lying on planes parallel to the x_1x_2-plane. The fibre shape and direction is completely determined by a unit plane vector $\boldsymbol{a}$, with non-zero components a_α (Greek indices take values 1 and 2); the material thus behaves as locally transverse isotropic, with $\boldsymbol{a}$ defining the local direction of transverse isotropy. The so-called n-curves are the orthogonal trajectories of the a-curves in the x_1x_2-plane; they have tangent vector $n = (n_1, n_2)^T = (-a_2, a_1)^T$ and are not material curves. Those two families of curves may form an alternative, local, rectangular, curvilinear co-ordinate system on the x_1x_2-plane. The assumed state of uniform plane stress suggests that the Cauchy stress tensor, $\boldsymbol{\sigma}$, is a plane tensor with non-zero components $\sigma_{\alpha\beta}$ only.

Consideration of fibre bending resistance necessitates introduction of a couple-stress vector, $\boldsymbol{m}$, with non-zero components $m_{\alpha 3}$ (Soldatos, 2009a, 2010a,b, 2012). Hence, unlike (Dvorak and Bahei-El-Din, 1987; Soldatos, 2011) where fibres are considered perfectly flexible, the present study postulates that the considered