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Optimal topological design through insertion and configuration of finite-sized heterogeneities

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ABSTRACT

In this paper, we develop a procedure for optimal topological design by sequentially inserting *finite-sized non-spherical inclusions or holes* within a homogeneous domain. We propose a new criterion for topology change that results in a trade-off problem to achieve the greatest/least change in the objective for the least/greatest change in the size of the inclusion/hole respectively. We derive the material derivative of the proposed objective, termed as the configurational derivative, that describes sensitivity of arbitrary functionals to arbitrary motions of the inclusion/hole as well as the domain boundaries. We specifically utilize the sensitivity to position, orientation and scaling of finite-sized heterogeneities to effect topological design. We simplify the configurational derivative to the special case of infinitesimally small spherical inclusions or holes and show that the developed derivative is a generalization of the classical topological derivative. The computational implementation relies on B-spline isogeometric approximations. We demonstrate, through a series of examples, optimal topology achieved through sequential insertion of a heterogeneity of fixed shape and optimization of its configuration (location, orientation and scale).

1. Introduction

Often in engineering practice, there is a need to perform optimal topological design by placing *finite-sized, regular-shaped* geometries within the structure. Classically, topological design of structures is achieved by optimally distributing material in a fixed region with known loading and boundary conditions (Bendsoe and Sigmund, 2003). One commonly applied strategy for topology optimization is to consider the material as having a varying density in the range $[0, 1]$, which at the lower limit results in a void. Another approach to topology optimization is through homogenization of periodic media, where the microstructure at each point in the design space is optimized (Bendsoe and Sigmund, 2003). Shape optimal design (see Pironneau, 1984; Bennett and Botkin, 1986), in contrast to topology optimization, is concerned with determining the optimal boundary shapes of (typically homogeneous) objects that satisfy criteria such as minimum mass. Efforts at integrating topology and shape optimization have often focused on automating the transition between them (see for example Lin and Chao, 2000; Tang and Chang, 2001; Ansola et al., 2002).

In practice, the efficiency of topology and shape optimal design procedures is strongly dependent on the availability of analytically derived domain and shape design sensitivities (Dems and Mroz, 1983, 1984; Haug et al., 1986; Sokolowski and Zolesio, 1992) and their implementation in a finite element code. Therefore, the derivation of these sensitivities is a critical aspect of topology and shape optimal design.

Instead of topological design by distributing material optimally within the domain, an alternative approach to effecting topological modifications in the literature is by introducing infinitesimal holes and subsequently optimizing their size and shape. In spirit, this approach resembles shape optimal design. The advantages of such an approach include procedural unification of topology and shape optimization, greater control over resulting topologies and shapes, an ability to handle geometrical constraints imposed by manufacturing process, and smaller number of design variables leading to greater computational efficiency.

In an early study, Eschenauer et al. (1994) developed the "bubble method," in which the conditions for introducing an infinitesimal hole into the structure was derived. The hole was subsequently parameterized using NURBS basis functions, the structure meshed using finite elements, analyzed, leading eventually to the optimized hole shape. This procedure was applied iteratively leading to a sequential procedure for topological modification. More recently, the bubble method has been extended as the "bubble-and-grain" method through conditions for introduction planar, elliptical, infinitesimal inclusions for the strain energy density objective (Kobelev, 2010).

The generalization of the bubble method for introduction of infinitesimal spherical holes in the domain is through the notion of topological derivative (Sokolowski and Zochowski, 1999; Cea

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et al., 2000). The topological derivative is an estimate of the change in shape functionals due to introduction of infinitesimal spherical holes in the interior of the domain. To overcome the assumptions on the nature of the cost functions and the boundary conditions imposed on the introduced holes, an alternative definition of topological derivative corresponding to an expanding spherical hole in the spirit of shape sensitivity analysis was introduced by Novotny et al. (2003). In the limit when the hole radius asymptotically approached zero, the alternative definition of sensitivity led to the usual definition of topological sensitivity as the sensitivity corresponding to creation of a hole. Further, the topological derivative has also been extended to introduction of infinitesimal ellipsoidal inclusions (Cedio-Fengya et al., 1998; Nazarov and Sokolowski, 2003; Ammari and Kang, 2005; Amstutz, 2006).

In general, the topological derivative does not lend itself to introduction and modification of *finite-sized heterogeneities*. There are very few studies that appear to have explored the effect of introducing finite-sized heterogeneities on *arbitrary functionals* defined over the domain. In Gopalakrishnan and Suresh (2008), the authors develop the notion of feature sensitivity wherein the effect of introducing a finite-sized hole parameterized using a scalar feature (or scaling) parameter in the range $[0,1]$ was estimated. The feature sensitivity was then used to explore the impact of finite-sized hole introduced at various locations within the domain, i.e., to enable "fast reanalysis." The procedure also necessitated a solution to an exterior boundary value problem of a finite-sized hole placed in an infinite domain.

Optimal topological design through introduction of finite-sized inclusions or holes into homogeneous domains will in general require sensitivities of *arbitrary functionals* to changes in position, orientation or scaling. Such sensitivities of arbitrary functionals appear to be uncommon in the literature, and seem to have been explored only in Dems and Mroz (1986) in the context of deriving conservation rules, or path independent integrals, within solids that are *homogeneous* except for a crack or a void. Although such conservation rules have played a critical role in the field of fracture mechanics (see for instance, Eshelby, 1956; Rice, 1968; Knowles and Sternberg, 1972; Budiansky and Rice, 1973), the sensitivities of arbitrary functionals to translation, rotation and scaling do not appear to have been heretofore exploited for optimal topological design.

Finally, from a numerical solution perspective, the need to remesh domains with evolving inclusion/hole shapes remains a significant challenge. Therefore, *numerical examples based on arbitrary topological modifications effected by insertion and growth of an explicitly defined heterogeneity* have been relatively few in the literature.

Based on the above survey of literature, the goal of this paper is to demonstrate optimal topological design through insertion and configuration of finite-sized holes and inclusions by

1. Identifying a criterion and its material time derivative that provides sensitivity to the configuration (location, orientation and scale) of both "soft" as well as "stiff" finite-sized inclusions.
2. Showing that simplification of the material derivative to infinitesimal, spherical inclusions results in the classical topological derivative.
3. Illustrating through a series of examples the approach to effecting optimal topology by sequentially inserting, orienting and scaling finite-sized inclusions and contrasting the resulting optimal design to those obtained by placing infinitesimal heterogenieties.

2. Configurational derivative and optimal location, orientation and scaling conditions

In this section, we derive the necessary conditions to determine the optimal configuration of an inclusion within a homogeneous domain whose boundaries evolve in time. In particular, we permit both the inclusion boundary as well as the underlying matrix boundary to evolve. The optimal configuration of the inclusion is determined by solving the problem described below.

2.1. The configuration optimization problem

Given a domain Ω we describe a "design transformation" that is continuous with the pseudo "design time" t within the domain such that

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

where $\mathbf{X}$ is initial position in the domain independent of time and Ω denotes the configuration at any time instant t. Also, Γ denotes the boundary of the domain Ω. We define Ω^{t_0} as the initial configuration. As with $\mathbf{X}, \Omega^{t_0}$ is assumed independent of time. Associated with this design "deformation," a "design velocity" may now be defined as:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \mathbf{x}}{\partial t} \tag{2}$$

We now generalize the above body by introducing a heterogeneity defined over Ω_p bounded by Γ_p (see Fig. 1) located at position $\mathbf{x}_p$ inside the domain Ω. We define an objective over this heterogeneous domain as:

$$f(t) = \int_\Omega \psi \, d\Omega \tag{3}$$

where $\psi \equiv \psi(\varepsilon(\mathbf{x}(t)), \mathbf{x}(t), t)$ is the value of the design criterion at instant t. Henceforth, we will suppress the arguments of ψ for ease of reading. The corresponding quantity in the homogeneous domain is ψ^0. We associate with the inclusion and outside of it densities $\rho(\mathbf{x}(t), t)$ and $\rho^0(\mathbf{x}(t), t)$ respectively. In other words, outside of the inclusion, the density in the domain is the same as that in the homogeneous domain. In general, we permit the inclusion to be either "stiff" ($\rho > \rho^0$ in Ω_p and $\psi < \psi^0$ in $\Omega - \Omega_p$) or "soft" ($\rho < \rho^0$ in Ω_p and $\psi > \psi^0$ in $\Omega - \Omega_p$).

The goal of the configuration optimization problem is to optimally determine the reference location $\mathbf{x}_p$ of the inclusion, the orientation $\mathbf{n}_p$ of a reference axis passing through $\mathbf{x}_p$, and a rotation θ about the reference axis as well as the inclusion shape to achieve the greatest/least "effect" for the least/greatest change in mass of a stiff/soft inclusion. Thus, we formally pose the *configuration optimization problem* as the following trade-off optimization problem: find $\mathbf{x}_p, \mathbf{n}_p, \theta$ and the optimized inclusion shape to

$$\begin{aligned} &\text{minimize} \quad g(t) = \pm \int_\Omega (\psi - \psi^0) d\Omega + w \int_\Omega (\rho - \rho^0) d\Omega \\ &\text{Subject to} \int_\Omega \varepsilon : \mathbf{C} : \varepsilon^a d\Omega - \int_\Gamma \mathbf{t} \cdot \mathbf{u}^a d\Gamma = 0 \\ &\qquad\qquad \int_\Omega \varepsilon^0 : \mathbf{C}^0 : \varepsilon^{a0} d\Omega - \int_\Gamma \mathbf{t}^0 \cdot \mathbf{u}^{a0} d\Gamma = 0 \end{aligned} \tag{4}$$

where the positive sign on the objective applies for a stiff inclusion, and the negative sign for a soft inclusion; $\varepsilon^a(\mathbf{x}(t))$ and $\mathbf{u}^a(\mathbf{x}(t))$ are compatible virtual strains and displacements respectively. Similarly, $\varepsilon^{a0}(\mathbf{x}(t))$ and $\mathbf{u}^{a0}(\mathbf{x}(t))$ are corresponding virtual quantities in the homogeneous domain; $\mathbf{t}$ and $\mathbf{t}^0$ are the tractions on the boundary of the domains (assumed unchanging with time) with and without the inclusion respectively. $\mathbf{C}$ and $\mathbf{C}^0$ are the fourth rank elasticity tensors (assumed constant with time) in the inhomogeneous and homogeneous domains. Implicit in the above statement is the fact that on the portion of the boundary Γ_u where displacement boundary conditions are applied, $\mathbf{u}^a = \mathbf{u}^{a0} = 0$. Γ_t is the portion of boundary where tractions are prescribed, and $\Gamma = \Gamma_u \cup \Gamma_t$. The body forces are ignored in the constraints corresponding to the principle of virtual work for convenience. We show that by imposing the virtual