International Journal of Solids and Structures 49 (2012) 2515–2527

Contents lists available at SciVerse ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Numerical methods for contact between two joined quarter spaces and a rigid sphere

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ARTICLE INFO

Article history:
Received 13 March 2012
Received in revised form 14 May 2012
Available online 7 June 2012

Keywords:
Joined quarter spaces
Equivalent inclusion method
Eigenstrain
Fast Fourier transform

ABSTRACT

Quarter space problems have many useful applications wherever an edge is involved, and solution to the related contact problem requires extension to the classical Hertz theory. However, theoretical exploration of such a problem is limited, due to the complexity of the involved boundary conditions. The present study proposes a novel numerical approach to compute the elastic field of two quarter spaces, joined so that their top surfaces occupy the same plane, and indented by a rigid sphere with friction. In view of the equivalent inclusion method, the joined quarter spaces may be converted to a homogeneous half space with properly established eigenstrains, which are analyzed by our recent half space-inclusion solution using a three-dimensional fast Fourier transform algorithm. Benchmarked with finite element analysis the present method of solution demonstrates both accuracy and efficiency. A number of interesting parametric studies are also provided to illustrate the effects of material combinations, contact location and friction coefficient showing the deviation of the solution from Hertz theory.

1. Introduction

The Hertz theory of contact is based on the assumption of a homogeneous half space. However, many engineering systems such as rail/wheel contact, roller bearing contact or contact of welded bodies pose a challenge to the extension of contact theories for more complex considerations. The contact analyses for quarter space related problems are of great importance in both theoretical research and practical applications. Hetenyi (1970) employed an iterative scheme to calculate the stress in an elastic quarter space subjected to a concentrated normal load on its surface. The key to this iterative scheme utilizes two overlapped symmetrically loaded half spaces; numerical methods are employed to free the interfacial plane from normal stress. A coupled pair of integral equations with an unknown pressure can be derived from the two overlapped half spaces. Keer et al. (1983) used the Fourier transform to solve the integral equations and extended the solution approach to an elastic quarter space subjected to tractions on its surface. Furthermore, Hanson and Keer (1990) used a direct method to solve the integral equations, in which arbitrary loading could be considered. Moreover, the contact behavior near the edge of a quarter space is complex, and the edge effects arising from a quarter space during contact were examined by numerical approaches (Hanson and Keer, 1991, 1995; Hanson et al., 1994) or experiments (Chai and Lawn, 2007; Gogotsi and Mudrik, 2009; Mohajerani and Spelt, 2010). On the other hand, the finite element method (FEM) was employed by Bower et al. (1987) to analyze the plastic deformation of a quarter space under rolling contact loads. The contents of the above literature were limited to problems associated to a single quarter space. The contact problems of two joined quarter spaces, i.e., welded materials, were not involved. A quarter space can be treated as a special case of two joined quarter spaces, by setting Young's modulus of one of the materials to be zero.

The present analysis uses the equivalent inclusion method (EIM) originally proposed by Eshelby (1957) to investigate the contact between a rigid sphere and two joined quarter spaces. One of them is regarded as an inhomogeneity, which is treated by an equivalent inclusion of the same material constants as the other (cf. Eshelby (1957) and Mura (1993)) but with proper eigenstrain distribution. The entire displacement or stress fields are obtained by the superposition of the homogeneous half space solutions and the disturbed solutions due to the equivalent eigenstrains. In numerical procedure, contact pressure is approximated as piecewise constant over rectangular patches. The homogeneous half space solutions for the elastic displacements or stresses caused by the uniform pressure distributed on a rectangular area were obtained by Love (1929), Johnson (1985), Kalker (1986), Ahmadi et al. (1987), Hills et al. (1993), Liu and Wang (2002).

In the proposed modeling, analytical solutions for inclusions can be obtained following the direction of many known works, such as those for the elastic fields in a half space caused by

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http://dx.doi.org/10.1016/j.ijsolstr.2012.05.027

Nomenclature

a	Hertzian contact radius, mm
C_{ijkl}, C^*_{ijkl}	elastic moduli for the matrix and inhomogeneity, respectively, MPa
$C^{u_z}_p, C^{u_z}_{q_x}, C^{u_z}_{q_y}$	the influent coefficients of pressure-displacement and shear traction-displacement
E_1	Young's modulus for region 1, GPa
E_2	Young's modulus for region 2, GPa
F	Galerkin vectors
g	surface gap, mm
h_0	body separation between two surfaces, mm
$\mathbf{I}$	unit matrix
M, N, L	grid numbers along x, y and z directions, respectively
M_y	moment about the y-axis, N · mm
p	pressure, MPa
p_h	maximum Hertzian pressure, MPa
q_x,q_y	shear tractions parallel to the x, y direction, respectively, MPa
R	sphere radius, mm
$T^{(0)}_{ijkl}, T^{(1)}_{ijkl}, T^{(2)}_{ijkl}, T^{(3)}_{ijkl}$	influence coefficients relating eigenstrain to stress
u_z	surface displacements in the z direction, $u_z = u^e_z + \tilde{u}_z$, where u^e_z and $\tilde{u}_z$ denote half space solution and disturbed solution due to inhomogeneity, respectively, mm
W	applied normal load, N
x, y, z	space coordinates, mm
$\mathbf{x}'$,$\mathbf{x}$	vectors of the source point and target point, respectively
δ_z	rigid displacement in the z direction, mm
δ_{ij}	Kronecker delta
Δx,Δy,Δz	grid sizes of x, y and z directions respectively, mm
ε^0_{ij}	initial strain
$\tilde{\varepsilon}_{ij}$	perturbed strain caused by inhomogeneity
ε^*_{ij}	eigenstrain
μ_f	friction coefficient
v_1	Poisson's ratio for the body 1
v_2	Poisson's ratio for the body 2
σ_{ij}	stress, MPa
σ^0_{ij}	stress due to pressure and shear tractions, MPa
σ^*_{ij}	eigenstress due to eigenstrain, MPa
μ	shear modulus, MPa
Ω	domain of inhomogeneity

Special symbols

$*$	convolution
:	tensor contraction between a fourth-rank tensor and a second-rank tensor
Tilde($\sim$) or FT	the Fourier transform
IFFT	inverse discrete fast Fourier transform

Subscripts

1, 2	left and right quarter spaces, respectively

spherical thermal inclusions (Mindlin and Cheng, 1950b), ellipsoidal inclusions (Seo and Mura, 1979), or cuboid inclusions (Chiu, 1978). Note that the eigenstrains given in the above references were assumed to be uniform. On the other hand, numerical approaches built upon these elementary solutions can make the analysis applicable to more general cases. The elastic fields due to eigenstrains can be expressed in terms of Galerkin vectors (Mindlin and Cheng, 1950a), where the basic Galerkin vectors in a half space were derived by Yu and Sanday (1991a,b). Based on the Galerkin vectors, Liu and Wang (2005) initiated the study of the stress fields due to eigenstrains in a half space; the full set of analytical solutions for the displacements and stresses were achieved in Liu et al. (2012). Furthermore, Liu et al. (2012) derived the influence coefficients in explicit closed-form for numerical implementation, when the computational domain is divided into a number of elementary cuboids with uniform eigenstrains. The resultant elastic field caused by all such eigenstrains was the sum of solutions contributed by each individual inclusion, where the computation highlights a seamless implementation of the three-dimensional fast Fourier transform (FFT) algorithms (Liu et al., 2000). The current work relies heavily on this method to accelerate the calculation.

The equivalent eigenstrains chosen to replace the inhomogeneities need to be determined in advance. Recently, Chen et al. (2010) and Zhou et al. (2011a,b) used the Conjugate Gradient Method (CGM) to solve the equivalent eigenstrains. Chen et al. (2010) used this method to analyze the elasto-plastic contact on a layered half space. In their model, the layered material was treated as an inhomogeneity. Furthermore, the contact problems of a single inhomogeneity or a stringer of inhomogeneities in a half space subjected to an indentation was investigated by Zhou et al. (2011a). On the other hand, spherical inhomogeneities (Leroux et al., 2010) and cylindrical inhomogeneities (Leroux and Nélias, 2011) were also investigated. However in their numerical approach, the methods used to solve the disturbed solutions due to eigenstrains were based on Zhou et al. (2009), which is an approximate method possessing difficulties in numerical error control. The computational domain needs to be extended significantly and the surface mesh should be fine enough to capture accurately the disturbance behavior of near surface inclusions. However, when a fine surface mesh is used, such an indirect method would experience extra difficulties in developing explicit relations between the redundant surface traction and the unknown eigenstrain. The present contact analysis of the two joined quarter spaces model is based on the EIM theories employing the explicit solutions of the eigenstress and the influence coefficients developed by Liu et al. (2012), which, in contrast, have circumvented the above mentioned numerical difficulties.

2. Theoretical description

2.1. Equivalent inclusion method for joined quarter spaces

The contact model for a rigid sphere indenting two joined quarter spaces is shown schematically in Fig. 1. The two quarter spaces, denoted as region 1 and region 2 with different material properties, are perfectly "welded" together. Region 2 can be treated as an inhomogeneity (denoted by Ω) with respect to the region 1. When the two joined quarter spaces are subject to an external load, elastic fields will be disturbed by the inhomogeneity. The inhomogeneity can be replaced by an inclusion using properly chosen eigenstrains (Eshelby, 1957). This method is called equivalent inclusion method (EIM). The stress field of the two joined quarter spaces, shown in Fig. 2 (a), can be express as follows, using Hooke's law,

$$\sigma_{ij} = C^*_{ijkl}\left(\varepsilon^0_{kl} + \tilde{\varepsilon}_{kl}\right) = C_{ijkl}\left(\varepsilon^0_{kl} + \tilde{\varepsilon}_{kl} - \varepsilon^*_{kl}\right) \quad \text{in } \Omega \tag{1a}$$

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon^0_{kl} + \tilde{\varepsilon}_{kl}\right) \quad \text{in D} - \Omega \tag{1b}$$

where C_{ijkl} and C^*_{ijkl} denote the elastic moduli of the matrix (D − Ω) and inhomogeneity (Ω), respectively; $\varepsilon^0_{kl}$is the homogeneous (i.e. in the absence of material inhomogeneity) contact solution caused by the surface pressure; and $\tilde{\varepsilon}_{kl}$ is the perturbed strain caused by the