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A semi-analytical method with a system of decoupled ordinary differential equations for three-dimensional elastostatic problems

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ABSTRACT

In this paper, a new semi-analytical method is presented for modeling of three-dimensional (3D) elastostatic problems. For this purpose, the domain boundary of the problem is discretized by specific subparametric elements, in which higher-order Chebyshev mapping functions as well as special shape functions are used. For the shape functions, the property of Kronecker Delta is satisfied for displacement function and its derivatives, simultaneously. Furthermore, the first derivatives of shape functions are assigned to zero at any given node. Employing the weighted residual method and implementing Clenshaw–Curtis quadrature, coefficient matrices of equations' system are converted into diagonal ones, which results in a set of decoupled ordinary differential equations for solving the whole system. In other words, the governing differential equation for each degree of freedom (DOF) becomes independent from other DOFs of the domain. To evaluate the efficiency and accuracy of the proposed method, which is called Decoupled Scaled Boundary Finite Element Method (DSBFEM), four benchmark problems of 3D elastostatics are examined using a few numbers of DOFs. The numerical results of the DSBFEM present very good agreement with the results of available analytical solutions.

1. Introduction

Numerical approaches are usually employed to solve elastostatic problems for analysis and design purposes. Various types of numerical approaches such as Finite Element Method (FEM), Boundary Element Method (BEM), and Scaled Boundary Finite Element Method (SBFEM), are commonly used to solve two- and three-dimensional (2D and 3D) elastostatic problems.

The use of the FEM is popular since its procedures are well-established and versatile in nature (see for example, Gan et al. (2005), Papanicolopulos et al. (2009), Rashid and Selimotic (2006), and Zienkiewicz and Taylor (2000), among others for solving 3D problems).

The BEM principally requires reduced surface discretizations, and may be regarded as an appealing alternative to the FEM for elastostatic problems (see for example, Banerjee and Henry (1992), Chen and Lin (2010), Cruse (1969), Denda and Wang (2009), Masters and Ye (2004), Milroy et al. (1997), Mittelstedt and Becker (2006), Pan and Yuan (2000), Turco and Aristodemo (1998), Wang and Denda (2007), and Wu and Stern (1991), for 3D problems). As the BEM requires no domain discretization, fewer unknowns are needed to be stored. The BEM needs a fundamental solution for the governing differential equation in order to derive the boundary integral equation. This means that the BEM requires fundamental solutions that are dependent on the problem of interest. Although coefficient matrices of the BEM are much smaller than those of the FEM, they are usually fully-populated, non-symmetric, and non-positive definite.

Combining the advantages of the BEM and the FEM, the SBFEM has been successfully developed by Wolf (2004). By transforming the governing partial differential equations to ordinary differential equations, the SBFEM discretizes only the domain boundary of interest with surface finite elements. The SBFEM, which does not require any fundamental solution as for the BEM, has also been used for the analysis of 3D elastostatic problems (see Doherty and Deeks (2003) and Song (2004) among others).

In addition to the above-mentioned numerical methods, some other analytical and semi-analytical methods have been presented to solve the 3D elastostatic problems. Gao and Rowlands (2000) have developed a new hybrid experimental-analytical/numerical method for stress analysis of finite 3D elastostatics problems. Li and Fan (2001) have considered 3D interface inclusion problem based on the representations of displacements and stresses in term of Love's strain potential and Hankel transform technique. Kucher and Markenscoff (2004) have formulated the boundary value problem of traction for inhomogeneous anisotropic elastic materials in terms of stresses and applied it to spherically anisotropic materials. Peng et al. (2005) have presented a new simple engineering method for estimating the stress-intensity factor

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around a quarter-cracks emanating from a notch. Vetyukov et al. (2011) have suggested a novel asymptotic approach to the theory of non-homogeneous anisotropic piezoelectric plates.

In this paper, using a new semi-analytical method, called hereafter Decoupled Scaled Boundary Finite Element Method (DSBFEM), 3D bounded and unbounded elastostatic problems are investigated. The DSBFEM is the extension of the previous research of the authors for solving 2D potential (Khaji and Khodakarami, 2011) and 2D elastostatic (Khodakarami and Khaji, 2011) problems. Therefore, the main concepts of the method are given in this paper. In other words, emphasis is mainly devoted to those important features of the DSBFEM which are subjected to considerable modifications compared to the previous works performed by the authors. In the DSBFEM, analogous to the previous works for solving 2D bounded potential and elastostatic problems, only domain boundaries are discretized. Consequently, the governing ordinary differential equations are solved in the problem domain, analytically. The elements of the domain boundary are of special subparametric type in which, new special shape functions and higher-order Chebyshev mapping functions are employed. Proposing a weighted residual form and using Clenshaw–Curtis quadrature, the coefficient matrices of the system of equations become diagonal, which results in decoupled ordinary differential equations for the whole system. This means that the governing equation for each degree of freedom (DOF) is independent from other DOFs of the domain boundary. Accuracy and efficiency of the DSBFEM are illustrated through four benchmark problems.

2. 3D elastostatic governing equations in global coordinates

The equilibrium equations in elasticity may be solved based on either a strong or a weak formulation of the problem. In the strong formulation, one may directly elaborate the equilibrium equations and associated boundary conditions (BCs) written in a differential form. In the weak formulation one uses an integral form of the equations of motion.

The DSBFEM is a semi-analytical procedure which is based upon a weak formulation of the governing equations of elastostatic problems. The equilibrium equations for a 3D domain Ω ($\Omega \subset \mathbb{R}^3$) shown in Fig. 1 may be described by

$$\sigma_{ij,j} + f_i = 0 \tag{1}$$

in which σ_{ij} indicates the stress tensor components, and f_i denotes the external source of exciting forces per unit volume. For a 3D domain in global coordinates, $i = X, Y, Z$ and $j = X, Y, Z$ (see Fig. 1(a)).

Instead of employing the governing equations and corresponding boundary conditions directly (i.e., the strong form of Eq. (1)), one may use a weak form (e.g. integral form as weighted residual method). This may be performed by weighting Eq. (1) with arbitrary weighting function (w_i), and integrating over the problem domain Ω. This results in the following form

$$\int_\Omega w_i(\sigma_{ij,j} + f_i)d\Omega = 0 \tag{2}$$

or

$$\int_\Omega w_i\sigma_{ij,j}d\Omega + \int_\Omega w_i f_i d\Omega = 0. \tag{3}$$

Eq. (3) will be followed and discussed in Section 5.

3. Geometry modeling by mapping functions

To analyze a problem using numerical methods, the problem domain should be discretized. In the DSBFEM, a scaling center (SC) is chosen from which all domain boundaries are visible (Fig. 1(a)). For the bounded domains, the SC can be selected inside the domain or on the boundary. As a result, the total boundary of the domain consists of two types of surfaces: the surface that pass through the SC, and the other remaining surfaces. In the DSBFEM, only the surfaces that does not pass through the SC should be discretized by n_e two-dimensional (2D) subparametric elements S_e^ξ, $e = 1, 2, \ldots, n_e$, so that $S^\xi = \cup_{e=1}^{n_e} S_e^\xi$ (see Fig. 1(b)).

In the DSBFEM, a geometry transmission from global Cartesian coordinates $(\widehat{x}, \widehat{y}, \widehat{z})$ to local dimensionless coordinates (ξ, η, ζ) is proposed. The transmission is performed by Chebyshev polynomials as mapping functions. Three dimensionless local coordinates ξ, η and ζ are defined as ξ is radial coordinate from the SC to the boundaries, while η and ζ are tangential coordinates on the boundary surfaces. The radial coordinate ξ is equal to zero at the SC and is equal to 1 on the boundary surfaces. The tangential coordinates η and ζ vary between -1 and $+1$ on the boundary surfaces.

In addition, the displacement and stress components at each node are interpolated by special shape functions that are introduced in this paper. The mapping functions and the special shape functions are illustrated in the following sections.

After discretizing the boundary surfaces, the domain boundary geometry is approximated using mapping functions. Each element on the boundary is analogous to a quadrilateral; thus, an appropriate one-by-one mapping between a square parent element and each real physical element S_e^ξ may be established. In the DSBFEM, subparametric elements whose mapping functions $[\Phi(\eta, \zeta)]$ are different from shape functions $[N(\eta, \zeta)]$ are introduced (see section 4 for more details on shape functions). If the global coordinates of the ith node of element S_e^ξ on the boundary surfaces are denoted by x_i, y_i and z_i, each element S_e^ξ may be defined in terms of a set of M mapping functions $\phi_a(\eta, \zeta)$ which is related to nodes a, $a = 1, 2, \ldots, M$. The geometry of elements in local coordinates is then written as

$$\{x(\eta, \zeta)\} = [\Phi(\eta, \zeta)]\{x\} \tag{4}$$

or,

$$x(\eta, \zeta) = \sum_{i=1}^{M} x_i\phi_i(\eta, \zeta), \quad y(\eta, \zeta) = \sum_{i=1}^{M} y_i\phi_i(\eta, \zeta),$$
$$z(\eta, \zeta) = \sum_{i=1}^{M} z_i\phi_i(\eta, \zeta) \tag{5}$$

in which

$$\{x(\eta, \zeta)\} = \lfloor x(\eta, \zeta), y(\eta, \zeta), z(\eta, \zeta)\rfloor^T,$$
$$\{x\} = \lfloor x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_M, y_M, z_M\rfloor^T \tag{6}$$

and $[\Phi(\eta)] = [\phi_1(\eta)[I], \phi_2(\eta)[I], \ldots, \phi_M(\eta)[I]]$, and $[I]$ indicates a 3×3 identity matrix. Furthermore, x, y and z denote the global coordinates of the boundary surface points.

In the DSBFEM, any point in the domain with $\widehat{x}, \widehat{y}$ and $\widehat{y}$ coordinates relates to the corresponding point on the elements of the boundary using the following equations

$$\widehat{x}(\xi, \eta, \zeta) = \xi x(\eta, \zeta) = \xi\sum_{a=1}^{M} x_a\phi_a(\eta, \zeta), \tag{7}$$

$$\widehat{y}(\xi, \eta, \zeta) = \xi y(\eta, \zeta) = \xi\sum_{a=1}^{M} y_a\phi_a(\eta, \zeta), \tag{8}$$

$$\widehat{z}(\xi, \eta, \zeta) = \xi z(\eta, \zeta) = \xi\sum_{a=1}^{M} z_a\phi_a(\eta, \zeta). \tag{9}$$

In order to produce mapping functions, the Chebyshev polynomials are employed. The number of nodes in each boundary element is denoted by $M = (n_\eta + 1)(n_\zeta + 1)$, where $(n_\eta + 1)$ and $(n_\zeta + 1)$ indicate the numbers of nodes along η or ζ directions,