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Homogenization of geomaterials containing voids by random fields and finite elements

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ABSTRACT

The paper describes the use of random fields and finite elements to assess the influence of porosity and void size on the effective elastic stiffness of geomaterials. A finite element model is developed involving "tied freedoms" that allows analysis of an ideal block of materials leading to direct evaluation of the effective Young's modulus and Poisson's ratio. The influence of block size and representative volume elements (RVE) are discussed. The use of random fields and Monte-Carlo simulations deliver a mean and standard deviation of the elastic parameters that lead naturally to a probabilistic interpretation. The methodology is extended to a foundation problem involving a footing on an elastic foundation containing voids. The approach enables estimates to be made of the probability of excessive settlement.

1. Introduction

The motivation for this work came from a study of foundations resting on subsurface materials containing voids of variable porosity and size (Griffiths et al., 2011a,b). Such sites may consist of a karst topography, which is a special type of landscape and subsurface characterized by the dissolution of soluble rocks, including limestone and dolomite. Even if the expected porosity of the site can be conservatively estimated, the location of the voids may be unknown lending itself to a probabilistic analysis. In addition, two sites with the same porosity may have quite different void sizes, where one has numerous small voids, while the other, fewer large voids. To facilitate modeling of boundary value problems, the goal of this work is to determine the effective elastic parameters of such materials, where the effective values are defined as the Young's modulus and Poisson's ratio (or shear and bulk modulus) that would have led to the same response if the material had been homogeneous.

In this paper, we use the random finite element method (RFEM) (e.g. Fenton and Griffiths, 2008) to examine the influence of voids on the parameters of an elastic material. The method starts with a conventional plane strain FE model of an elastic block of material, after which a random field of values is generated taking account of local averaging (e.g. Fenton and Vanmarcke, 1990) and mapped onto the mesh. The goal of the study is to generate results giving guidance on the mean and standard deviation of the effective Young's modulus and Poisson's ratio as a function of porosity and void size. The parametric studies reported in this paper also give insight into the relationship between the representative volume element (RVE) for a material containing voids and the number of Monte-Carlo simulations needed to reach statistical convergence.

The void volume and size within the specimen is controlled though parameters of the random field as will be explained in the next two sections. Having established the statistical distributions of effective properties as mentioned above, the information can then be applied to more practical boundary value problems. Later in this paper, we consider the influence of voids on the settlement of a strip footing, leading to estimates of the probability of excessive settlement.

The behavior of a heterogeneous material with micro-structure, consisting of varying properties has been studied by a number of investigators. The goal is to obtain the effective or equivalent properties at the macro-scale. An important objective of micromechanics is to link mechanical relations going from finer to coarser length scales.

It is assumed that the stiffness parameters of the intact material (e.g. E *and* υ) are known, and the goal of the investigation then becomes one of assessing the macro-stiffness of the material when it is interspersed with voids. A useful concept in this homogenization

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Nomenclature

A	element area	ε_y	normal strain in y direction
B	footing width	δ_x	horizontal deformation
C	settlement proportionality constant	δ_y	vertical deformation
E	effective Young's modulus	δ_υ	vertical deformation in settlement analysis
E_i	effective Young's modulus at the ith simulation	$\delta_{\upsilon i}$	vertical deformation at the ith simulation
E_0	Young's modulus of intact material	θ	spatial correlation length (dimensional)
Δx	element width	Θ	spatial correlation length (non-dimensional)
Δy	element height	υ	effective Poisson's ratio
L	width and height of block	μ_{E/E_0}	mean of effective normalized Young's modulus
n	porosity	σ_{E/E_0}	standard deviation of effective normalized Young's modulus
$P[\cdot]$	probability	μ	mean
Q	vertical force	μ_υ	mean of υ
x, y	cartesian coordinates	ρ	correlation coefficient
Z	random variable	σ, σ^2	standard deviation, variance
$z_{n/2}$	value of the standard normal variable	$\sigma^2_{(A)}$	variance after local averaging
α	dimensionless element size parameter	σ_υ	standard deviation of υ
γ	variance reduction due to local averaging	τ_x, τ_y	difference between x and y coordinates of two points
σ_x	normal stress in x direction	$\Phi[\cdot]$	standard normal cumulative distribution function
σ_y	normal stress in y direction		
σ_z	normal stress in z direction		
ε_x	normal strain in x direction		

process is the representative volume element or RVE. An RVE is an element of the heterogeneous material that is large enough to capture the effective properties in a reproducible way. From a modeling point of view, the smallest RVE that can achieve this is of particular interest (e.g. Liu, 2005).

The concept of the RVE was first introduced by Hill (1963), since when there have been many numerical simulations developed and applied to determine RVE size (e.g. Kulatilake, 1985; Kanit et al., 2003; Ning et al., 2008; Esmaieli et al., 2010; Huang et al., submitted for publication). Several theoretical models have also been proposed for dealing with scale effects ranging from micro to macro levels. The Differential Method (Roscoe, 1952) has been one of the most effective and widely used methods. The Composite Spheres Model (Hashin, 1962) considered only a single inclusion and led to simple closed-form expressions. The Self Consistent Method (Budiansky, 1965; Hill, 1965) and the Generalized Self Consistent Method, formalized by Christensen and Lo (1979) involved embedding an inclusion phase directly into an infinite medium. Christensen and Lo (1979) explained that the final form of this method can solve the spherical inclusion problem. Finally, the Mori and Tanaka (1973) method as described by Benveniste (1987) has attracted a lot of interest and involves quite complex manipulations of the field variables along with special concepts of strain and stress. Although there are many analytical models for estimating the effective elastic properties of a material containing voids, they are often limited to voids with simple shapes. See also the review of Klusemann and Svendsen (2009).

Numerical methods such as the finite element method (FEM) or the boundary element method (BEM) have been used to validate some of the theoretical approaches. Two major variables can be investigated in a realistic representation of a defective material; namely the volume and size of the voids or inclusions. Isida and Igawa (1991) considered several kinds of periodic arrays of holes, while Day et al. (1992) considered a material containing circular holes within a triangular or hexagonal matrix and occasionally over-lapping random circular holes. Hu et al. (2000) developed a numerical model based on BEM to estimate effective elastic properties such as Young's modulus, bulk modulus and shear modulus. The main objective was to investigate the influence of the shortest distance between holes of random size and volume based on a normal distribution. Cosmi (2004) introduced a new numerical model called the Cell Method (CM) to investigate the effect of randomly located voids. The model consisted of a homogeneous matrix of cells which contains randomly located voids. Li et al. (2010) developed an FEM model to calculate the elastic properties of porous materials with randomly distributed voids.

2. Finite element model

Assuming consistent units, the initial finite element mesh for this study (e.g. Smith and Griffiths, 2004) considers a square plane strain block of material modeled by 50×50 8-node square elements of unit side length ($\Delta x = \Delta y = 1$) as shown in Fig. 1. The boundary conditions allow vertical movement only of nodes on the left side, horizontal movement only of nodes on the bottom side, with the bottom-left corner node fixed. The vertical components of all nodal freedoms on the top loaded side are "tied", as are the horizontal components of all nodal freedoms on the right side. Tied freedoms are forced to move by the same amount in the analysis because they are assigned the same freedom number during stiffness assembly. The tied freedom approach offers an elegant way of modeling a heterogeneous medium as an ideal element of material. The tied freedom approach ensures that the square deforms into a rectangle. Other methods employing stress or strain control may give similar outcomes, but the proposed tied freedom approach, while resulting in neither uniform stresses nor strains within the block, allows an exact back-calculation of equivalent elastic parameters as will be described.

A vertical force shown as $Q = 50$ in the figure is applied to the tied vertical freedom on the top of the square imposing an average unit vertical pressure of $Q/L = 1$. The boundary conditions ensure that no matter what degree of heterogeneity is introduced, such as, for example, the darker regions in Fig. 1 indicating voids, the mesh will always deform as an ideal element with the top surface remaining horizontal and the right side remaining vertical. From