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Numerical study of the wrinkling of a stretched thin sheet

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ABSTRACT

A thin sheet clamped at opposite ends and stretched develops wrinkles parallel to the direction of the applied tensile strain due to the hindered Poisson lateral contraction at the clamps. To study this phenomenon, a variational model recently proposed by Puntel, Deseri and Fried is adopted. The relevant energy functional includes bending and membranal contributions and is minimized subject to a constraint on the area of the mid-surface of the sheet. A fourth order partial-differential equation is henceforth obtained and numerically implemented using B-splines. Predictions are obtained concerning the number of wrinkles, critical applied stretches, and scaling relationships for wrinkle amplitude and wavelength. Both a linearized version of the boundary-value problem based on the small-slope approximation and a fully nonlinear one are considered: their results are found to be in good agreement for the whole range of applied stretches taken into account. Comparisons with previous analytical results by Puntel, Deseri and Fried, who used different boundary conditions and an Ansatz on the deflection function are also provided. The numerical results substantially confirm the validity of the analytical predictions. The present work provides then an alternative numerical method for the study of wrinkling in thin sheets and supports the use of analytical and semi-analytical solutions as viable options for specific geometries. Though further investigation, particularly experimental, is still needed, extensive comparisons of the results with other studies available in the literature provide confirmation for the scaling laws and signal that predicted values of the critical stretches may only be accurate for higher length-to-width aspect ratios.

1. Introduction

Wrinkles arise in thin sheet subject to in-plane loading as manifestations of instability due to the almost negligible flexural stiffness of membranes and the associated inability to withstand compressive loading.

In general, a membrane subject to in-plane loading involving a combination of shear and tension will generally exhibit taut, wrinkled, and slack zones. In taut zones, the in-plane principal stresses are tensile. In wrinkled zones, one of the in-plane stresses vanishes and the crests and troughs of the wrinkles are roughly parallel to trajectories of nonvanishing tension. In slack zones, which do not appear in the absence of wrinkles, the in-plane principal stresses both vanish.

Wrinkles have traditionally and firstly been a matter of concern in the aerospace industry for their negative impact on the performance and longevity of thin structural elements. To avoid wrinkles and to apply tension as uniformly as possible throughout the membrane surface, compensating supports such as pillars and arches are introduced or cables are attached to the edges and interior of the membrane. This solution is not always feasible because it involves often unwelcome weight increases and because larger tensile stresses can intensify risks of material creep, crack initiation and propagation, and failure. When otherwise wrinkles can or have to be accounted for, the stress distribution changes significantly and can be computed resorting to tension-field theory (see e.g. Steigmann, 1990). Albeit elegant and mathematically sound, this theory neglects the bending stiffness of the membrane and assumes wrinkles of infinitesimal amplitude and wavelength. It is therefore not helpful when aiming at characterizing wrinkles. This can occur, as observed by Jenkins et al. (1999), when attempting to ensure that precision requirements of space reflector antennas are met.

Leaving aside aerospace applications, there are several other areas in which quantitative assessment of wrinkling is of current importance. In the steel lamination industry different wrinkling patterns occur during rolling and leveling, i.e. straightening by stretching, of thin metal strips. Differences in the distribution of plastic strains along the width of the strip are inferred from the observation of wrinkles (Fischer et al., 2000). The analytical and numerical work by Fischer et al. (2000) is also relevant for the similarity of the problem studied therein with the one investigated here. In the medical field, there is a need for wound and burn

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treatments (Hudson and Renshaw, 2006) and for surgical techniques that ensure proper circulation, mobility and minimal scarring subsequent to healing (Lott-Crumpler and Chaudhry, 2001; Georgeu and Ross, 2002; Cerda, 2005; Hofer and Mureau, 2009; Ueda et al., 2009). Optical measurements of wrinkles on thin polymer films might provide a simple and accurate alternative to more traditional techniques used to determine thicknesses, mechanical properties (Stafford et al., 2004), and residual stresses (Chung et al., 2009). Similarly, measuring wrinkles created by cells crawling on thin silicone substrates might provide important insight regarding cell locomotion (Burton and Taylor, 1997; Harris et al., 1980). Applications that exploit wrinkling to tune the optical properties of cavities (Kolaric et al., 2010) and to shape capillaries for micro-fluidic purposes (Ohzono et al., 2009) also seem relevant.

The first numerical approaches to wrinkling were based on membrane elements and tension-field theory. One such approach, called *Iterative Material Properties* (IMP), was initially developed by Miller and Hedgepeth (1982) and subsequently by Miller et al. (1985). An iteratively and element-wisely varying Poisson's ratio is introduced to model the geometric strain in the direction perpendicular to the wrinkles. In this approach, an ad hoc *wrinkling criterion* based on the combined evaluation of the sign of the principal stresses and strains is used to activate the modified material properties.

Other tension-field theory approaches using membrane elements are based on iterative no-compression models devised by Contri and Schrefler (1988).

Liu et al. (2000) employed the tension-field theory solution obtained using membrane elements to derive information on actual wrinkling patterns via semi-analytical determination of impeding buckling modes.

In general, however, shell elements are preferred when aiming at the qualitative characterization of wrinkles. The first to employ three-dimensional shell elements to this end were Tomita and Shindo (1988). A common trait to studies adopting shell elements is the application of suitable imperfections to trigger the out-of-plane deflection modes when starting from an in-plane stretched membranal configuration. Wong and Pellegrino (2006c) used preliminarily extracted eigenmodes of the tangent stiffness matrix as initial imperfections. Other alternatives include randomly distributed out-of-plane imperfections, as proposed by Tessler et al. (2005), and the application of small forces perpendicular to the membrane, as advocated by Leifer and Belvin (2003).

The present study was motivated by the experimental and analytical work of Cerda et al. (2002) and Cerda and Mahadevan (2003). In their experiment, a polyethylene thin sheet of rectangular shape is clamped along two opposite edges and stretched. As a result of the hindered Poisson contraction occurring near the fixed ends, a zone of in-plane compressive stresses forms and wrinkling ensues. An effective depiction of this phenomenon can be found in Friedl et al. (2000). Cerda et al. (2002) measured the wavelength, but not the amplitude, of the wrinkles and the corresponding stretch at which they occurred.

The analytical part of Cerda, Ravi-Chandar and Mahadevan's work has two distinguishing features. The first distinguishing feature lies in the energy functional that is minimized to solve the boundary-value problem for the stretched thin sheet. In addition to terms that can be likened to membranal and Föppl–von Kármán bending energy contributions, a geometric constraint on the length of the sheet in the direction perpendicular to the direction of stretch is introduced. Cerda, Ravi-Chandar and Mahadevan consider pinned boundary conditions at the clamped ends which do not reproduce the hindered Poisson contraction effect. Hence, the pre-stretch of the thin sheet is a homogeneous uniaxial tension. For this reason, the geometric constraint is precisely an ad hoc measure that forces the thin sheet to explore out-of-plane wrinkled configurations afar from the otherwise stable flat solution. The second distinguishing feature is an educated guess regarding the structure of the deflected solution. Specifically, it is assumed to be a half-sinusoid in the direction of stretch and a sinusoid depending on three parameters, namely amplitude, wavelength and phase, in the direction perpendicular to the direction of stretch. Minimizing of the constrained energy functional yields scaling relations for the amplitude and wavelength of wrinkles as functions of the applied stretch. The former scaling relation fits well with the experimental data. No critical values of the applied stretch associated with the onset of wrinkled configurations are provided.

In a recent paper, Puntel et al. (2011) reconsider the work of Cerda, Ravi-Chandar and Mahadevan. Both the variational framework and the pinned boundary conditions used in that work are retained. However, each term of the energy functional is revised. The geometric constraint is relaxed by prescribing the area of the mid-surface of the sheet. Membranal and bending energy terms are properly derived from kinematical assumptions. Moreover, no Ansatz is made for the deflection in the direction perpendicular to the direction of stretch. Minimization of the energy functional leads to a differential Euler–Lagrange equation and no longer to a set of algebraic equations. To solve the problem analytically, the Euler–Lagrange equation is further linearized according to a small-slope approximation.

Numerous results are obtained on this basis. First, a sequence of values of applied stretch is determined, each corresponding to the onset of a configuration with a different number of wrinkles. Second, wrinkle amplitude and wavelength are expressed as functions of the applied stretch. For sufficiently large values of the applied stretch, the scaling relations obtained by Cerda et al. (2002) are recovered. Finally, wrinkles are found to have unequal amplitude in the direction perpendicular to the applied stretch, namely decreasing in a semi-sinusoidal fashion from the center of the sheet towards the free edges.

The present study constitutes a numerical counterpart of the paper of Puntel et al. (2011). Its main aim is to remove several of the simplifying assumptions motivated by the analytical treatment, thus obtaining improved solutions and allowing for an assessment of the validity range of the simplifications imposed previously. In particular, fixed, in contrast to pinned, boundary conditions are imposed at the fixed edges. Moreover, at the free edges, exact traction-free boundary conditions are used instead of phenomenological ones. Of course, no a priori Ansatz is adopted for the deflection. Two different cases are investigated: one in which the small-slope approximation, which enables the linearization of the Euler–Lagrange equation, is retained and one in which it is removed. In either case, the geometric constraint is nonlinear and iterative solution schemes are required. A finite-element code, based on B-splines, has been developed specifically for this study together with the weak formulation of the variational problem and the tangent stiffness matrix for the Newton–Raphson algorithm. The main outcome of the analyses is that, despite the several simplifications removed, for the specific geometry considered the analytical results obtained more directly by Puntel et al. (2011) are essentially still valid. The present work provides then an alternative numerical method for the study of wrinkling in thin sheets and supports the use of analytical and semi-analytical solutions as viable options for specific geometries. Though further investigation, particularly experimental, is still needed, extensive comparisons of the results with other studies available in the literature provide confirmation for the scaling laws and signal that predicted values of the critical stretches may only be accurate for higher length-to-width aspect ratios.

The paper is structured as follows. In Section 2, the boundary-value problem is formulated. In Section 3, the results of the work of Puntel et al. (2011) are summarized. In Section 4, the role of the geometric constraint in the variational formulation is discussed.