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Transient wave analysis of a cantilever Timoshenko beam subjected to impact loading by Laplace transform and normal mode methods

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ABSTRACT

This study applies two analytical approaches, Laplace transform and normal mode methods, to investigate the dynamic transient response of a cantilever Timoshenko beam subjected to impact forces. Explicit solutions for the normal mode method and the Laplace transform method are presented. The Durbin method is used to perform the Laplace inverse transformation, and numerical results based on these two approaches are compared. The comparison indicates that the normal mode method is more efficient than the Laplace transform method in the transient response analysis of a cantilever Timoshenko beam, whereas the Laplace transform method is more appropriate than the normal mode method when analyzing the complicated multi-span Timoshenko beam. Furthermore, a three-dimensional finite element cantilever beam model is implemented. The results are compared with the transient responses for displacement, normal stress, shear stress, and the resonant frequencies of a Timoshenko beam and Bernoulli–Euler beam theories. The transient displacement response for a cantilever beam can be appropriately evaluated using the Timoshenko beam theory if the slender ratio is greater than 10 or using the Bernoulli–Euler beam theory if the slender ratio is greater than 100. Moreover, the resonant frequency of a cantilever beam can be accurately determined by the Timoshenko beam theory if the slender ratio is greater than 100 or by the Bernoulli–Euler beam theory if the slender ratio is greater than 400.

1. Introduction

The dynamic transient response of a beam is an important topic in engineering applications. Although the Bernoulli–Euler beam theory (classical beam theory) is most widely used, it has no upper bound for the wave velocity and overestimates the natural frequencies. Moreover, the Bernoulli–Euler beam theory provides accurate results for slender beams rather than for thick beams. Timoshenko (Timoshenko, 1921, 1922) improved the beam theory by including the influence of shear and rotary inertia. Therefore, the Timoshenko beam theory not only has upper bounds for wave velocities but also agrees well with the natural frequencies and mode shapes of the exact two-dimensional theory (Fung, 1965; Graff, 1973; Labuschagne et al., 2009). Consequently, the Timoshenko beam theory is more appropriate for analyzing transient responses, especially in situations involving high frequency vibrations and thick beams. Stephen and Puchegger (2006) made a comparison of the resonant frequencies of bending vibration of a short free beam to test the valid frequency range of Timoshenko beam theory.

In this study, the Laplace transform method and the normal mode method are employed to investigate the transient response of a Timoshenko cantilever beam subjected to impact loading. The Laplace transform method can be classified into two approaches for inverse transformation: theoretical and numerical inverse approaches. Although the theoretical inversion is able to yield the exact solution (ray solution), the integration in a complex plane is difficult, and the numerical calculation time is extensive. From this perspective, the numerical Laplace inversion is needed because inverse transforms can be obtained more easily and efficiently. Abundant literature is available that discusses the methods of numerical inversion of Laplace transformation, and they can be classified into four groups (Davies and Martin, 1979). The first group contains methods that represent the function using polynomials. This group contains several mathematical approaches such as Legendre polynomials (Papoulis, 1956), Jacobi polynomials (Max et al., 1966), Chebyshev polynomials (Lanczos, 1957), and Laguerre polynomials (Weeks, 1966). The second group contains methods based on Gaussian quadrature formulas (Piessens, 1970). The third group is a method of trapezoidal integration along a special integral contour (Talbot, 1979). Duffy compared three numerical methods of the Laplace inversion and concluded that Talbot proposed an optimum parameter selection method (Duffy, 1993). The fourth group is comprised of methods based on Fourier

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series. Dubner and Abate used the Fourier cosine transformation to perform the numerical Laplace inversion (Dubner and Abate, 1968), and Durbin (1974) improved it by including the Fourier sine transformation into the Dubner and Abate method. As a result, the numerical error in the Durbin method is independent of time and valid for the whole period of the series. Crump (1976) proposed a method based on Dubner and Abate but which converged more quickly. Honig and Hirdes (1984) made an improvement to reduce the dependence of discretization and truncation errors on the free parameters. Because the methods based on the Fourier series have an excellent accuracy on a wide range of functions (Davies and Martin, 1979), the Durbin method is used in this study to evaluate transient responses of the Timoshenko beam.

The normal mode method (i.e., mode superposition or eigenfunction expansion), which expresses a transient response by superposing all the steady state responses, can provide a long-time response for numerical calculation. Traill-Nash and Collar (1953) presented the frequency equations and mode shapes for three types of end supports and compared them with experimental values. Anderson and Pasadena (1953) solved the transient response for a simply supported beam problem. Han et al. (1999) analyzed the steady state and transient responses for the Bernoulli–Euler, Rayleigh, shear, and Timoshenko beams. Van Rensburg and Van der Merwe (2006) discussed natural frequencies and mode shapes of the Timoshenko beam in detail. Su and Ma (2011) investigated the dynamic response of a simply-supported Timoshenko beam by ray and normal mode methods. Although many investigations of the normal mode method can be found, very few papers presented the results in close form solutions, which is significant for the efficiency of the numerical calculation for the normal mode method. This study provides the close form solutions of the normal mode method for the cantilever Timoshenko beam and discusses the numerical results with the Durbin method. The methodology proposed by Ma's group (Lee and Ma, 1999; Ma et al., 2001; Ma and Lee, 2006) for solving a multi-layered media problem is successful, and the Durbin method provides the greatest promise of inversing the Laplace transformation (Davies and Martin, 1979). These two formulations are integrated to solve dynamics problems of complex structures.

This paper is organized as follows. Section 2 presents the solutions in the Laplace transform domain and the transient responses are obtained by the Laplace inverse transformation base on the Durbin method. In Section 3, the normal mode method is employed to analyze the Timoshenko cantilever beam subjected to impact loadings. Based on these two approaches, the comparison of the transient responses for displacement, shear force and bending moment is made in Section 4. The normal mode method (theoretical method) is used as a standard for a convergence check for the Laplace transform and Durbin method (numerical method). Furthermore, the comparisons of resonant frequencies and transient responses for displacement, normal stress and shear stress base on the Bernoulli–Euler beam, Timoshenko beam and ABAQUS FEM are discussed in this section. Finally, a conclusion is made in Section 5.

2. Transient solutions based on the Laplace transform method

As shown in Fig. 1, a cantilever beam is considered, and the left endpoint of the beam is denoted as node [1], while the right endpoint of the beam is node [2]. The origin of the coordinate x is set at node [1]. The beam with length L is subjected to an interior impact force $F_0H(t)$ at $x = d$, where $H(t)$ is the Heaviside function. The transient responses of the cantilever beam will be derived and discussed by the Laplace transform method in this section and the normal mode method in the next section.

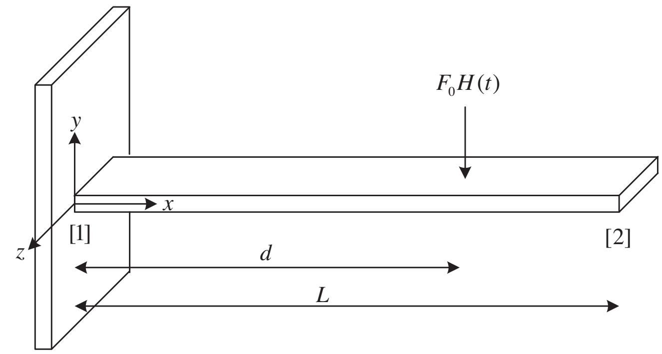

Fig. 1. Configuration of the cantilever beam and the dynamic impact force.

2.1. Solution in the transform domain

Based on the Timoshenko beam theory, the equations of motion for a beam can be written as

$$\begin{cases} \kappa GA\frac{\partial^2 y_s}{\partial x^2} = \rho A\frac{\partial^2(y_s+y_b)}{\partial t^2}, \\ EI\frac{\partial^3 y_b}{\partial x^3} + \kappa GA\frac{\partial y_s}{\partial x} = \rho I\frac{\partial^3 y_b}{\partial x\partial t^2}, \end{cases} \tag{1}$$

where E is Young's modulus, ρ is the material density, A is the cross-sectional area of the beam, I is the cross-sectional moment of inertia, G is the shear modulus, κ is the shear coefficient, and y_b and y_s denote the transverse displacements due to bending moment and shear force, respectively. The transverse displacement is expressed as

$$y(x,t) = y_b(x,t) + y_s(x,t). \tag{2}$$

The bending slope of deflection curve, shear force, and bending moment are given, respectively, by

$$\phi = \frac{\partial y_b}{\partial x}, \quad V = \kappa AG\frac{\partial y_s}{\partial x}, \quad M = EI\frac{\partial^2 y_b}{\partial x^2}. \tag{3}$$

The initial conditions are presented as

$$y_b(x,0) = y_s(x,0) = \frac{\partial y_b(x,0)}{\partial t} = \frac{\partial y_s(x,0)}{\partial t} = 0. \tag{4}$$

The boundary conditions at $x = 0$ and $x = L$ are as follows

$$y^{[1]} = y(0,t) = 0, \quad \phi^{[1]} = \frac{\partial y_b(0,t)}{\partial x} = 0, \tag{5a}$$

$$M^{[2]} = M(L,t) = EI\frac{\partial^2 y_b(L,t)}{\partial x^2} = 0, \quad V^{[2]} = V(L,t) = \kappa AG\frac{\partial y_s(L,t)}{\partial x} = 0. \tag{5b}$$

We applied the Laplace transform over time t with transform parameter p for boundary conditions in the transform domain. The Laplace transform of an arbitrary function $f(x,t)$ is defined by

$$F(x;p) \equiv \int_0^\infty f(x,t)e^{-pt}dt, \tag{6}$$

where p is a positive real number, large enough to ensure the convergence of the integral. By using the Laplace transform, the governing Eq. (1) become two coupled ordinary differential equations as follows

$$\begin{cases} \kappa GA\frac{d^2\hat{y}_s}{dx^2} = \rho Ap^2(\hat{y}_s + \hat{y}_b), \\ EI\frac{d^3\hat{y}_b}{dx^3} + \kappa GA\frac{d\hat{y}_s}{dx} = \rho Ip^2\frac{d\hat{y}_b}{dx}. \end{cases} \tag{7a}$$

Substituting $\hat{y}_b = H(p)\exp(\lambda x)$ and $\hat{y}_s = L(p)\exp(\lambda x)$ into Eq. (7a) yields