International Journal of Solids and Structures 49 (2012) 2121–2137

Contents lists available at SciVerse ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

A gradient elasticity theory for second-grade materials and higher order inertia

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ARTICLE INFO

Article history:
Received 4 November 2011
Received in revised form 23 March 2012
Available online 26 April 2012

Keywords:
Gradient elasticity
Higher order inertia
Continuum thermodynamics
Dynamics
Wave dispersion

ABSTRACT

Second-grade elastic materials featured by a free energy depending on the strain and the strain gradient, and a kinetic energy depending on the velocity and the velocity gradient, are addressed. An *inertial energy balance principle* and a *virtual work principle for inertial actions* are envisioned to enrich the set of traditional theoretical tools of thermodynamics and continuum mechanics. The state variables include the *body momentum* and the *surface momentum*, related to the velocity in a nonstandard way, as well as the concomitant *mass-accelerations* and *inertial forces*, which do intervene into the motion equations and into the force boundary conditions. The boundary traction is the sum of two parts, i.e. the *Cauchy traction* and the *Gurtin–Murdoch traction*, whereas the traction boundary condition exhibits the typical format of the equilibrium equation of a material surface (as known from the principles of surface mechanics) whereby the Gurtin–Murdoch traction (incorporating the inertial surface force) plays the role of applied surfacial force density. The body's boundary surface constitutes a *thin boundary layer* which is in global equilibrium under all the external forces applied on it, a feature that makes it possible to exploit the traction Cauchy theorem within second-grade materials. This means that a second-grade material is formed up by two sub-systems, that is, the bulk material operating as a classical Cauchy continuum, and the thin boundary layer operating as a Gurtin–Murdoch material surface. The classical linear and angular momentum theorems are suitably extended for higher order inertia, from which the local motion equations and the moment equilibrium equations (stress symmetry) can be derived. For an isotropic material featured by four constants, i.e. the Lamé constants and two length scale parameters (Aifantis model), the dynamic evolution problem is characterized by a Hamilton-type variational principle and a solution uniqueness theorem. Closed-form solutions of the wave dispersion analysis problem for beam models are presented and compared with known results from the literature. The paper indicates a correct thermodynamically consistent way to take into account higher order inertia effects within continuum mechanics.

1. Introduction

The elastic materials considered in the present paper belong to the class of generalized polar and nonpolar materials studied by Truesdell and Toupin (1960), Toupin (1962), Mindlin (1964, 1965), Mindlin and Eshel (1968), Green and Rivlin (1964), and the micropolar materials addressed by Eringen (1966). However, for simplicity of exposition, we shall limit ourselves to considering second-grade materials, that is, the first strain gradient materials addressed by Mindlin (1964), Mindlin and Eshel (1968). More precisely, we shall follow the so-called Form-II formulation by the latter authors, whereby the higher order strain tensor is defined as the first gradient of the standard (second order) strain tensor, and the resulting stress tensors exhibit some useful symmetry properties (to be specified shortly). The interest for this class of materials stems from the possibility of associating to them higher order inertia effects, that is, the effects produced over such a material whenever it is in motion while the kinetic energy depends on the velocity gradient. This combination makes it possible to dispense with strain singularities at sharp crack tips and to capture some size effects within the material dynamic behavior, typically the wave dispersion phenomena manifested by real materials as polymer foams, high-thoughness ceramics, high-strength metal alloys, porous materials and the like, Mindlin (1964), Papargyri-Beskou et al. (2009) and Askes and Aifantis (2011).

Higher order inertia effects were considered in a paper by Mindlin (1964) dealing with elastic materials with microstructure, in which the Hamilton principle is employed to derive the relevant force balance equations and the material constitutive equations. On setting equal to zero the relative motion of the microstructure with respect to the continuum, Mindlin's theory can be shown to reduce itself to an elasticity theory in which the strain energy depends on the strain and the strain gradient, and the kinetic energy depends on the velocity and the velocity gradient. Mindlin (1964) showed the importance of the velocity gradient for the motion equations to be able to capture wave dispersion phenomena

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http://dx.doi.org/10.1016/j.ijsolstr.2012.04.019

(typical of nonhomogeneous materials). Mindlin's theory proves to be rather cumbersome for the excessive number of material constants needed (eighteen for isotropic materials with microstructure, seven in the case of form-II materials).

Germain (1973) also addressed materials with microstructure with inherent higher order inertia effects, but his intent was only to establish the pertinent balance equations by means of an *ad hoc* extended-form *principle of virtual power*.

More recently, the role of the velocity gradient and of the inherent higher order inertia terms in the motion equations has been systematically addressed in a series of studies dealing with the wave motion and the related dispersion phenomena. There exists a rich literature on this subject, but we limit ourselves to mention Altan and Aifantis (1997), Georgiadis et al. (2000), Askes et al. (2002), Askes et al. (2007), Metrikine and Askes (2002), Askes and Aifantis (2006, 2009) and Papargyri-Beskou et al. (2009), the review paper by Askes and Aifantis (2011) and the literature therein. From the latter group of papers, it emerges clearly how the higher order inertia models are able to describe realistically wave dispersion phenomena. Metrikine and Askes (2002), Askes and Aifantis (2006) and Askes and Aifantis (2009) advanced the concept of "dynamically consistent" gradient model, namely a model endowed with gradient enhancements in both its stiffness and inertia features, which renders it capable to remove singularities of the strain field in the presence of, for instance, a sharp crack tip, and to realistically describe the dispersive characteristics of the wave propagation in a nonhomogeneous medium. The higher order inertia terms appearing in the governing equations are there introduced heuristically by analogy to similar terms related to strain gradient problems, and their relationship to the kinetic energy remains unclarified.

Fried and Gurtin (2006) addressed second-grade materials in which the strain gradient and the velocity gradient engender, respectively, higher order stresses and higher order inertial forces. The principle of virtual power (PVP) is used to derive the pertinent balance equations for the internal and external force system, including the inertial body and surface forces there arising. Also, a nonstandard "inertial virtual power balance" law is devised as an extension to higher order inertia of an analogous law advanced by Podio-Guidugli (1997) for classical kinetic energy. The latter law, involving the kinetic energy as a function of the velocity and the velocity gradient, is used by Fried and Gurtin (2006) for the evaluation of the inertial forces in terms of acceleration and acceleration gradient. Although applicable also to solid materials, this theory seems to be mostly oriented towards fluid mechanics.

The present paper provides, within the framework of small deformations, a gradient elasticity theory for continua featured by a strain energy depending on the strain and the (first) strain gradient, as well as by a kinetic energy depending on the velocity and the (first) velocity gradient, that is, a theory in which the effects of both strain gradient and higher order inertia are combined. The main purpose of the paper is to ascertain the correct thermodynamically consistent way to take into account the higher order inertia effects within solid mechanics.

For this purpose, as a preliminary step, some fundamental notions of thermodynamics are presented in Section 2 and cast in a form suitable for subsequent extensions to second-grade materials exhibiting higher order inertia effects. In analogy to Podio-Guidugli (1997) and Fried and Gurtin (2006), an *inertial energy balance principle* is introduced, which parallels the classical *energy balance principle* (first thermodynamics principle). With a terminology borrowed from Noll (1963), we can state that the latter principle is associated to the actions (stresses, noninertial forces) arising from the exterior bodies belonging to our near world (as the solar system), whereas the former one is instead associated to the actions (momentum, inertial forces) arising from the totality of bodies belonging to the remote universe (i.e. the so-called fixed stars). These two principles are distinct from each other because, on one hand, the energy balance principle has to be invariant under change of observer; on the other hand, the inertial energy balance principle is concerned with quantities (as the velocity and the kinetic energy) that must be evaluated with respect to a Galilean observer, that is, one being fixed, or moving uniformly, with respect to the fixed stars (Noll, 1963; Truesdell and Noll, 1965).

The mentioned extension is realized in Section 3 for a class of second-grade thermo-elastic materials endowed with a kinetic energy being a quadratic function of the velocity and the first velocity gradient. Besides the usual stress and higher order stress tensors, a momentum and a higher order momentum are introduced with the cumulative name of *(generalized) local momenta*. The material constitutive equations for both sets of stresses and momenta are evaluated in a particular case of linear isotropic elasticity. The resulting elasticity model identifies with the well-known Aifantis model (Aifantis, 1992; Ru and Aifantis, 1993; Altan and Aifantis, 1997), characterized by four material constants, i.e. the two Lamé constants and two length scale parameters, one (ℓ_s) related to strain gradient effects, the other (ℓ_d) to higher order inertia.

In Section 4, the equilibrium equations for the noninertial forces are derived by means of a specific *principle of virtual power* (PVP). Although the latter principle is well known from the literature, (see e.g. Germain (1973), Gurtin (2001) and Fried and Gurtin (2006)), it is here discussed in detail considering the existence of singularities (edge lines, corner points) and assuming that the body and surface external forces cumulate the inertial ones. It is found that the well-known formula of the boundary traction, $\mathbf{t}$, for second-grade materials, whereby $\mathbf{t}$ depends not only on the normal $\mathbf{n}$ of the boundary surface, but also on the related mean curvature K, can be split into two distinct parts, say $\mathbf{t} = \mathbf{t}_C + \mathbf{t}_{GM}$. Here, $\mathbf{t}_C$ is the *Cauchy traction*, that is, the traction associated to the relevant (total) stress, $\mathbf{T}$, through the relation $\mathbf{t}_C = \mathbf{n} \cdot \mathbf{T}$, it thus depends only on the normal $\mathbf{n}$, whereas the remaining part, $\mathbf{t}_{GM} = \mathbf{t} - \mathbf{t}_C$ (*Gurtin–Murdoch traction*), depends on both $\mathbf{n}$ and K. Then, known notions of surface mechanics (Gurtin and Murdoch, 1975, 1978) are invoked to interpret the mentioned traction equation as an equilibrium equation for the boundary surface viewed as a material thin boundary layer, where $\mathbf{t}_{GM}$ plays the role of surfacial body force. It is also found that the external actions applied upon the thin boundary layer (including the forces acting on the edge lines and the corner points, if any) satisfy global equilibrium conditions similar to the global equilibrium equations for the whole body. As a consequence, the latter equilibrium equations of the body simplify such as to include only the body forces and the Cauchy traction, just like for a standard Cauchy continuum, and thus the Cauchy theorem for the traction can be applied also within the present context. It thus emerges that a second-grade material constitutes a combination of two co-operating structural parts, i.e. a classical Cauchy continuum (the bulk material) featured by the (Cauchy) stress $\mathbf{T}$ and the traction $\mathbf{t}_C = \mathbf{n} \cdot \mathbf{T}$, as well as a Gurtin–Murdoch material surface (the thin boundary layer) featured by a surface stress $\boldsymbol{\Sigma}$ and the traction $\mathbf{t}_{GM}$ as a surfacial body force.

In Section 5 a nonstandard *principle of virtual work* (PVW), specifically devoted to inertial actions, is formulated and applied to determine the equilibrium equations relating the generalized local momenta to the inertial forces. It is found that the latter momenta contribute to the formation of a *body momentum*, $\mathbf{p} := \rho\bar{\mathbf{v}}$, where $\bar{\mathbf{v}} := \mathbf{v} - \ell_d^2 \Delta \mathbf{v}$ is a (weak) nonlocal (or gradient enhanced) velocity, and a *surface momentum*, $\mathbf{p}_S := \ell_d^2 \rho \partial_n \mathbf{v}$. It is also found that the inertial forces substantiate as *inertial body force*, $\mathbf{b}^{\mathrm{in}}$, distributed within the bulk material, and *inertial surface force*, $\mathbf{t}^{\mathrm{in}}$, distributed over the boundary surface of the material, and that these inertial forces are equal to the negative body and surface *mass-accelerations*, that is, $\mathbf{b}^{\mathrm{in}} := -\boldsymbol{\varphi} = -\rho\dot{\bar{\mathbf{v}}}$ and $\mathbf{t}^{\mathrm{in}} := -\boldsymbol{\varphi}_S = -\ell_d^2 \rho \partial_n \dot{\mathbf{v}}$.