International Journal of Solids and Structures 48 (2011) 1474–1484

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Homogenization and dimensional reduction of composite plates with in-plane heterogeneity

Chang-Yong Lee, Wenbin Yu *

Department of Mechanical and Aerospace Engineering, Utah State University, Logan, UT 80322-4130, USA

ARTICLE INFO

Article history:
Received 21 October 2010
Received in revised form 28 January 2011
Available online 18 February 2011

Keywords:
Plate
Homogenization
Dimensional reduction
Variational asymptotic method
Heterogeneous plates

ABSTRACT

The variational asymptotic method is used to construct a new model for composite plates which could have in-plane heterogeneity due to both geometry and material. We first formulate the original three-dimensional problem in an intrinsic form which is suitable for geometrically nonlinear analysis. Taking advantage of smallness of the plate thickness and heterogeneity, we use the variational asymptotic method to rigorously construct an effective plate model unifying a homogenization process and a dimensional reduction process. This approach is implemented in the computer code VAPAS using the finite element technique for the purpose of dealing with realistic heterogeneous plates. A few examples are used to demonstrate the capability of this new model.

1. Introduction

Along with the rapidly increasing popularity of composite materials and structures, research on accurate and general modeling of structures made of them has remained as a very active field in the last several decades. Moreover the increased knowledge and fabrication techniques of them are possible to manufacture new materials and structures with optimized microstructures to achieve the ever-increasing performance requirements. Although it is logically sound to use the well-established finite element analysis (FEA) to analyze such materials and structures by meshing all the details of constituent microstructures, it is not a practical and efficient way, which requires an inordinate number of degrees of freedom (i.e., computing cost) to capture the micro-scale behavior.

Fortunately, most composite materials exhibit statistical homogeneity (Hashin, 1983) so that we can define a representative volume element (RVE), which is entirely typical of the whole mixture on average and contains a sufficient number of inclusions for the apparent overall properties to be effectively independent of the boundary conditions (Hill, 1963). Although different definitions are given for an RVE in the literature (Nemat-Nasser and Hori, 1993), we give a practice-oriented definition for an RVE as any block of material the analyst wants to use for the micromechanical analysis to find the effective properties and replace it with an equivalent homogeneous material. The term unit cell (UC) is also used extensively in the literature and defined as the building block of the heterogeneous material. In our work, we define UC as the smallest RVE. In other words, one RVE could contain several UCs. These definitions essentially imply that it is the analyst's judgement to determine what should be contained in an RVE or UC. To be consistent with statistical homogeneity, a well-formulated micromechanics model should not depend on the size of an RVE, which means the effective properties obtained from an RVE containing multiple UCs should be the same as those obtained from a UC. In this sense, we consider the heterogeneous structure as a periodic assembly of many UCs.

If the size of UC (d) is much smaller than the size of the structure (L) (i.e., $\eta = d/L \ll 1$), it is possible to homogenize the heterogeneous UC with a set of effective material properties through a micromechanical analysis of the UC. With these effective properties, the analyst can replace the original heterogeneous structure with a homogeneous one and carry out structural analysis for global behavior. In the past several decades, numerous micromechanical approaches have been suggested in the literature, such as the self-consistent model (Hill, 1965; Dvorak and Bahei-El-Din, 1979; Accorsi and Nemat-Nasser, 1986), the variational approach (Hashin and Shtrikman, 1962; Milton, 2001), the method of cells (Aboudi, 1982, 1989; Paley and Aboudi, 1992; Williams, 2005), recursive cell method (Banerjee and Adams, 2004), mathematical homogenization theories (Bensoussan et al., 1978; Sanchez-Palencia, 1980; Murakami and Toledano, 1990), finite element approaches using conventional stress analysis of a representative volume element (Sun and Vaidya, 1996), variational asymptotic method for unit cell homogenization (VAMUCH) (Yu, 2005; Yu and Tang, 2007), and many others (see, e.g. Hollister and Kikuchi (1992), Kalamkarov et al. (2009), Kanouté et al. (2009) for a review).

* Corresponding author. Tel.: +1 435 7978246; fax: +1 435 7972417.
E-mail address: wenbin.yu@usu.edu (W. Yu).

0020-7683/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2011.01.032

In real applications, many composite structures are dimensionally reducible structures (Yu, 2002) with one or two dimensions much smaller than others. For example, many load bearing components are flat panels with the thickness h much smaller than the in-plane dimensions (i.e., $e = h/L \ll 1$) and they can be effectively modeled using plate models. If there are still many unit cells along the thickness direction (i.e., $\eta \ll e$), we can use the traditional two-step approach that performs homogenization using micromechanics first to obtain effective properties of the heterogeneous material, then performs a dimensional reduction to construct a plate model for structural analysis. Usually, composite plates do not have many unit cells along the thickness direction. For example, for plates made of textiles, the textile microstructure might be as large as the plate thickness. That is, the periodicity is exhibited only in-plane and we have either $e \ll \eta$ or $e \sim \eta$. As pointed out by Kohn and Vogelius (1984), if $e \ll \eta$, the order of the aforementioned two-step approach should be reversed. That is, we need to carry out the dimensional reduction to construct plate models first, then homogenize the heterogeneous surface with periodically varying plate properties. If $e \sim \eta$, the two steps in the two-step approach should be performed at the same time, that is, both small parameters (e and η) should be considered during modeling of such structures. And several studies have shown that models considering e and η simultaneously also give accurate results for the case $e \ll \eta$ (Lewiński, 1991; Buannic and Cartraud, 2001).

In recent years, the formal asymptotic method has been used to study this problem (Caillerie, 1984; Kohn and Vogelius, 1984; Lewiński, 1991; Kalamkarov, 1992; Kalamkarov and Kolpakov, 1997). It is a modification to the asymptotic homogenization method which is a direct application of the formalism of two scales to the original three-dimensional (3D) equations governing the plate structure. However, although these models are mathematically elegant and rigorous without introducing *ad hoc* assumptions, it is not easy to relate the equations derived using this method with simple engineering models and extend this approach to geometrical nonlinear problems. Sometimes, the displacement field predicted using this approach is not compatible with the stress field. For example, the displacement field in Eqs. (1.3.5) of Kalamkarov and Kolpakov (1997) implies zero transverse normal strain which further implies nonzero normal stress due to Poisson's effect, which is not compatible with the stress field given in Eq. (1.3.6) of Kalamkarov and Kolpakov (1997). Last but not least, it is difficult to implement these theories numerically.

As a remedy to the shortcomings of formal asymptotic method, we propose to use the variational asymptotic method (VAM) (Berdichevsky, 1979) to carry out simultaneous homogenization and dimensional reduction to construct a model suitable for plates made of heterogeneous materials. First, the 3D anisotropic elasticity problem is formulated in an intrinsic form suitable for geometrically nonlinear analysis. Then, considering both e and η, we use VAM to rigorously decouple the original 3D anisotropic, heterogeneous problem into a nonlinear two-dimensional (2D) surface analysis (i.e., plate analysis) on the macroscopic level and a linear micromechanical analysis. The micromechanical analysis is implemented in the computer code VAPAS (Variational Asymptotic Plate and Shell analysis) using the finite element technique for numerically obtaining the effective plate constants for the 2D plate analysis and recovering the local displacement, strain, and stress fields based on the macroscopic behavior. Several examples are used to demonstrate the application and accuracy of this new model and the companion code VAPAS.

2. Three-dimensional formulation

A plate may be considered geometrically as a smooth 2D reference plane ω surrounded by a layer of matter with thickness h to form a 3D body with one dimension much smaller than the other two. In general, a point in the plate can be represented mathematically by its Cartesian coordinates x_i, where x_α are two orthogonal lines in the reference plane and x_3 is the normal coordinates. (Here and throughout the paper, Greek indices assume values 1 and 2 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated.) Without loss of generality, we choose the middle of the plate as the origin of x_3. Let us now consider an heterogeneous plate formed by many UCs (Ω) in the reference plane (see Fig. 1). To describe the rapid change in the material characteristics in the in-plane directions, we need to introduce two so-called 'fast' coordinates y_α parallel to x_α. These two sets of coordinates are related as $y_\alpha = x_\alpha/\eta$.

If the UC is a cuboid as depicted in Fig. 1, we can describe the domain (Ω) occupied by the UC using y_α and x_3 as

$$\Omega = \left\{ (y_1, y_2, x_3) \middle| -\frac{d_1}{2} < y_1 < \frac{d_1}{2}, -\frac{d_2}{2} < y_2 < \frac{d_2}{2}, -\frac{h}{2} < x_3 < \frac{h}{2} \right\} \tag{1}$$

As our goal is to homogenize the heterogenous material, we need to assume that the exact solution of the field variables have volume averages over Ω. For example, if $u_i(x_1, x_2, x_3; y_1, y_2)$ are the exact displacements within the UC, there exists $v_i(x_1, x_2)$ such that

$$v_i = \frac{1}{\Omega} \int_{y_1} \int_{y_2} \int_{x_3} u_i \, dy_1 \, dy_2 \, dx_3 = \frac{1}{\Omega} \int_\Omega u_i \, d\Omega \equiv \langle u_i \rangle \tag{2}$$

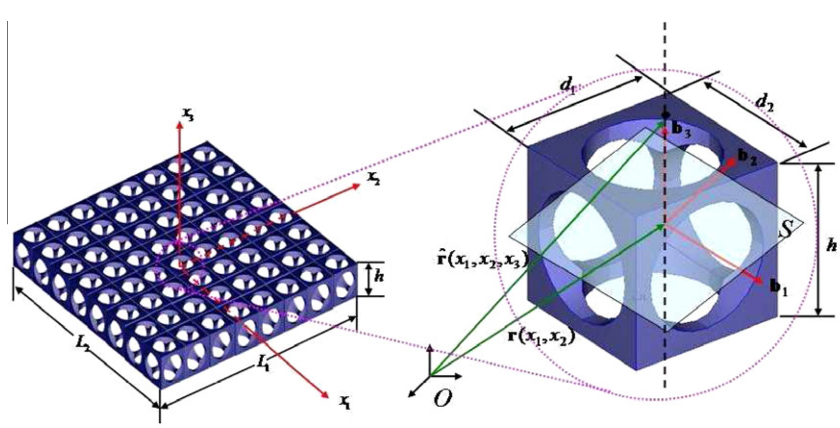

Fig. 1. A heterogeneous plate with representative periodicity cell.