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Optimal anisotropic three-phase conducting composites: Plane problem

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ABSTRACT

The paper establishes tight lower bound for effective conductivity tensor K_* of two-dimensional three-phase conducting anisotropic composites and defines optimal microstructures. It is assumed that three materials are mixed with fixed volume fractions and that the conductivity of one of the materials is infinite. The bound expands the Hashin–Shtrikman and translation bounds to multiphase structures, it is derived using a combination of translation method and additional inequalities on the fields in the materials; similar technique was used by Nesi (1995) and Cherkaev (2009) for isotropic multiphase composites. This paper expands the bounds to the anisotropic composites with effective conductivity tensor K_*. The lower bound of conductivity (G-closure) is a piece-wise analytic function of eigenvalues of K_*, that depends only on conductivities of components and their volume fractions. Also, we find optimal microstructures that realize the bounds, developing the technique suggested earlier by Albin et al. (2007a) and Cherkaev (2009). The optimal microstructures are laminates of some rank for all regions. The found structures match the bounds in all but one region of parameters; we discuss the reason for the gap and numerically estimate it.

1. Introduction

The problem. The paper investigates the lower bound for effective conductivity and optimal micro-geometries of three-material composites (plane problem). We assume that two mixing isotropic materials have finite conductivities k_1 and k_2 $(0 < k_1 < k_2)$ and the third one is a superconductor $k_3 = \infty$, the volume fractions $m_1 \geqslant 0$, $m_2 \geqslant 0$ and $m_3 = 1 - m_1 - m_2 \geqslant 0$ of the materials are fixed. The conductivity of a composite is characterized by an anisotropic effective conductivity tensor K_* that depends on the properties of mixed materials and their volume fractions, as well as on microstructures. We describe the bounds of *G-closure* (Lurie and Cherkaev, 1981) – the set of all effective properties of composites with arbitrary microstructure. The G-closure boundary depends only on k_1, k_2, m_1, and m_2. Optimal microstructures *realize the bound* if their effective conductivity lies at the G-closure boundary.

We find the bound solving a variational problem of minimization of K_* with respect to microstructures (Section 2). Namely, we apply two orthogonal external fields of different magnitudes to a periodic composite and minimize the sum J of the corresponding energies of the composite, varying the microstructure occupying the periodicity cell Ω. The computed value of J allows for computation of the add outer bound of G-closure, as discussed in Sections 2.2 and 2.3. The matching microstructures (minimizing sequences) are found by a different technique that was introduces in Albin et al. (2007a) and Cherkaev (2009) and is described in Sections 3.3 and 6; by assumption, optimal structures are laminates of some rank. The effective properties of the structures form the inner bound of G-closure. When the outer and inner bounds coincide, they are exact and the G-closure is determined. We show that our bounds are exact in all domains of parameters but one. In the last domain, we estimate the gap between the outer and inner bounds.

Remark 1.1. The complementary upper bound can be established by a solution of a dual problem, in which conductivity k_i are replaced by resistivity $\rho_i = 1/k_i$. In the considered problem, one of the component is a superconductor $(k_3 = \infty)$ which makes the dual bound trivial - the effective resistivity can be arbitrary large, or $K_*^{-1} \geqslant 0$. The obtained results allows for the upper bound determination for the G-closure of materials with conductivities $k_1 = 0 < k_2 < k_3 < \infty$.

Bounds. The problem of exact bounds has a long history. It started with the bounds by Voigt (1928) and Reuss (1929), called also Wiener bounds or the arithmetic and harmonic mean bounds. The bounds are valid for all microstructures and become in a sense exact for laminates: One of the eigenvalues of K_* of a laminate is equal to the harmonic mean of the mixed materials' conductivities, and the other one – to the arithmetic mean of them. The pioneering paper by Hashin and Shtrikman (1963) found the bounds and the

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matching structures for optimal isotropic two-component composites, and suggested bounds for multicomponent ones. The exact bounds and optimal structures of *anisotropic* two-material composites were found in earlier papers (Lurie and Cherkaev, 1982, 1986; Kohn and Strang, 1983, 1986; Tartar, 1985) using a version of the translation method (see its description in books (Cherkaev, 2000; Allaire, 2001; Milton, 2002; Dacorogna, 2008)). The method is equivalent to building the polyconvex envelope of a multiwell Lagrangian, as it was shown by Kohn and Strang (1983, 1986); the wells are the energy of the materials plus their cost (here, "cost" is the dual variable to the volume fraction of material in the composite). The theory of bounds for the two-material composite is now well developed and applied to elastic, viscoelastic, and other linear materials, see for example the books (Lurie, 1993; Cherkaev, 2000; Allaire, 2001; Milton, 2002; Dacorogna, 2008).

Bounds for multicomponent composites turn out to be much more difficult. Milton (1981) showed that the Hashin–Shtrikman bound is not exact everywhere (it tends to an incorrect limit when $m_1 \to 0$), but is exact when m_1 is larger than a threshold, $m_1 \geqslant g_m$. Milton and Kohn (1988) suggested an extension of the translation method to anisotropic multimaterial composites, computed the anisotropic bounds for multicomponent composites and the optimal structures. Nesi (1995) suggested a new tighter bound for *isotropic* multicomponent structures, and (Cherkaev, 2009) further improved it and found optimal structures. The method is based on the procedure suggested by Nesi (1995) that combines the translation method and additional inequality constraints (Alessandrini and Nesi, 2001). These two latest bounds coincide in the case $k_3 = \infty$ that is considered here.

This paper extends these bounds to *anisotropic* composites. As in the early paper by Kohn and Strang (1983), we investigate the case when one of the phase has infinite conductivity, which significantly simplifies the calculation. The method is based on constructing a lower bound for the composite energy accounting for Alessandrini and Nesi (2001) constraints. Because of the constraints, the translated energies-wells can become nonconvex but are still bounded from below, an improved bound corresponds to this case. The method is described in Section 3, the results are summarized in Section 4. The energy bound turns out to be a multifaced piece-wise analytic function of the problem's parameters. Like the translation bound, it depends only on conductivities of the materials, their volume fractions, and the anisotropy of a homogeneous external loading. The energy bounds and related bounds for the G-closure are derived in Section 5.

Optimal structures. In the paper, we prove that multiscale laminates realize the G-closure bound. Similar structures – laminates of second rank – realize the G-closure bound for the two-material case (Lurie and Cherkaev, 1982, 1986); three-material bound is achievable by more complex structures of the same kind. Optimal structures depend on the degree of anisotropy of the external loading.

Remark 1.2. Generally, optimal structures are not necessary laminates: for example, Hashin and Shtrikman (1963) first suggested "coated spheres" geometry, Milton (1981) introduced parallel coated spheres and later suggested (Milton, 2002) a method of transformation of optimal shapes, (Lurie and Cherkaev, 1985) suggested multilayer coated circles, Vigdergauz (1989) and Grabovsky and Kohn (1995) and recently (Liu, 2008) suggested special convex oval-shaped inclusions, Gibiansky and Sigmund (2000) suggested "bulk blocks", Albin and Cherkaev (2006) proved the optimality of "haired spheres", and recent paper by Benveniste and Milton (2003) investigated "coated ellipsoids". All these structures admit separation of variables when effective properties are computed. It is not clear yet if the laminate structure approximates any other optimal structure, see for example Pedregal (1997), Briane and Nesi (2004) and Albin et al. (2007b). We show, however, that proper laminates are optimal for the considered problem.

The topology of two-material optimal structures is simple and intuitively clear: for an isotropic or moderately anisotropic loading, the less conducting material k_1 "wraps" the more conducting one k_2 ($k_2 > k_1$), so that k_2 forms an nucleus and k_1 forms a core. If the anisotropy of the loading exceeds a threshold, the optimal structures degenerate into simple laminates.

The multimaterial structures are more diverse and nonunique and require new ideas for constructing. Milton (1981), Lurie and Cherkaev (1985) and later Barbarosie (2001) described two types of isotropic structures that realize the multicomponent bound for sufficiently large volume fractions $m_1 \geqslant g_m$ of the weaker conductor $k_1 < k_2 < \cdots$ Later, Sigmund (2000) and Gibiansky and Sigmund (2000) expand the domain of applicability of Hashin–Shtrikman bounds to $m_1 \geqslant g_{gs}$ where the threshold g_{gs} is smaller than the one previously known $g_{gs} < g_m$. They demonstrated new isotropic non-laminate microstructures (bulk structures) that realize this bound. Liu (2008) found another structures an optimal conductivity. Albin et al. (2007a) extended the results of Gibiansky and Sigmund (2000)) finding anisotropic laminates that realize translation bounds for both isotropic and anisotropic structures in a range of parameters

$$m_1 \geqslant g_{acn}, \quad \text{and } \frac{|k_{*2} - k_{*1}|}{k_{*1} + k_{*2}} \leqslant \hat{g}_{acn},$$

where k_{*1} and k_{*2} are eigenvalues of K_*. These inequalities restrict the range of volume fractions and degree of anisotropy of a composite that correspond to translation bounds. For isotropic composites ($k_{*1} = k_{*2}$), the range of applicability of the founded laminates coincides with the one of bulk structures founded by Gibiansky and Sigmund (2000). Structures that realize the isotropic bound for the whole range of volume fractions were found in Cherkaev (2009).

In Section 6 we extent this result to anisotropic composites, finding new optimal structures that realize our new bounds. More exactly, we show that optimal laminates realize the bounds in all but one region. The topology of optimal structures depends on volume fractions of the mixing elements and loading anisotropy level. The structure adjusts itself to meet the *sufficient* optimality conditions that are found during derivation of the bounds. All the optimal microstructures are found by the same procedure suggested in Albin et al. (2007a) and based on (i) the energy bounds and sufficient optimality conditions for gradient fields inside each material, and (ii) the lamination technique that allows for satisfaction of these conditions. In all cases but one, the found laminate achieve the bounds, they are not unique.

In the remaining case, the lower bound for G-closure is definitely not exact, hence the mentioned technique for building the structures is not applicable. In that case, we guess the best structures (that correspond to the upper bound of G-closure) basing on asymptotic behavior of optimal structures in neighboring regions and then numerically compute the gap between the structures and bound that is between the upper and lower bounds for G-closure). The gap is very small, see Section 6.4, which shows that the suggested laminates (upper bound) and the lower bound accurately approximate G-closure.

2. The problem

2.1. Equations and notations

Consider a periodic composite formed by three materials. The materials k_i occupy plane domains Ω_i, $i = 1,2,3 \subset R_2$ that form a unit periodicity cell Ω