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A Bending-Gradient model for thick plates. Part I: Theory

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ABSTRACT

This is the first part of a two-part paper dedicated to a new plate theory for out-of-plane loaded thick plates where the static unknowns are those of the Kirchhoff–Love theory (3 in-plane stresses and 3 bending moments), to which six components are added representing the gradient of the bending moment. The new theory, called the Bending-Gradient plate theory is described in the present paper. It is an extension to arbitrarily layered plates of the Reissner–Mindlin plate theory which appears as a special case of the Bending-Gradient plate theory when the plate is homogeneous. However, we demonstrate also that, in the general case, the Bending-Gradient model cannot be reduced to a Reissner–Mindlin model. In part two (Lebée and Sab, 2011), the Bending-Gradient theory is applied to multilayered plates and its predictions are compared to those of the Reissner–Mindlin theory and to full 3D Pagano's exact solutions. The main conclusion of the second part is that the Bending-Gradient gives good predictions of both deflection and shear stress distributions in any material configuration. Moreover, under some symmetry conditions, the Bending-Gradient model coincides with the second-order approximation of the exact solution as the slenderness ratio L/h goes to infinity.

1. Introduction

Laminated composite plates are widely used in engineering applications, especially in aeronautics. They offer excellent stiffness and strength performance for a low density. However, as fiber reinforced composites are very anisotropic materials, the overall plate properties of these laminates has been really difficult to capture. Because of a strong demand from industry for reliable models, many suggestions have been made.

Let us recall some essential requirements for such a model. The main goal is to simplify a computationally heavy 3D model into a 2D plate model without losing local 3D fields' accuracy. One would expect:

1. Good estimation of macroscopic deflection,
2. No limitation on local material symmetries,
3. A plate theory which is easy to implement with standard finite element tools,
4. Good relocalization of 3D fields in order to estimate local stresses.

The simplest and most widely-used theory is the Kirchhoff–Love plate model. This model is easy to implement and gives good estimates for in-plane stress components (far from the edges of the plate) and neglects the contribution of out-of-plane stress components to the stress energy. However, when the plate slenderness ratio L/h (h is the plate thickness and L the span) is not large enough, out-of-plane stresses have an increasing influence on the plate deflection. This phenomenon becomes sensitive when $L/h < 10$ for an isotropic plate and $L/h < 40$ for classical carbon fiber reinforced laminated plates and cannot be neglected for conventional use of composite laminates.

In recent decades many suggestions have been made to improve both deflection estimation and field localization for highly heterogeneous laminates. Reddy (1989), Noor and Malik (2000), and Carrera (2002) provided detailed reviews for these models. Two main approaches can be found: asymptotic approaches and axiomatic approaches. The first one is mainly based on the fact that h/L is a small parameter. Using asymptotic expansions in the small parameter h/L (Caillerie, 1984; Lewinski, 1991a,b,c), it is found that the Kirchhoff–Love kinematic is actually the first order of the expansion. However, higher-order terms yield only intricated "Kirchhoff–Love" plate equations and no simple model to implement. This difficulty is illustrated in Boutin (1996) for 3D periodic composites and in Buannic and Cartraud (2001a,b) for periodic beams. Another asymptotic method is based on the so-called Variational Asymptotic Method (VAM) applied to plates by Yu et al. (2002a,b). The strength of this approach is that it does not make more assumption than having h/L small and, according to its authors, it could be applied also to any non-linearities. However, this method does not seem simple to implement in conventional finite element code.

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The second main approach is based on assuming *ad hoc* displacement or stress 3D fields and often referred to as axiomatic approach. One of the assets of these approaches is that they seem easier to implement in finite element codes. These models can be "Equivalent Single Layer" or "Layerwise".

Equivalent Single Layer models treat the whole laminate as an equivalent homogeneous plate. Stress or displacement approaches have been suggested. Reissner (1945) was the first one who suggested a stress approach for homogeneous and isotropic plates. His approach will be detailed further in the present work. Reissner's transverse shear stress field is a parabolic distribution through the thickness. However, experiments and some exact solutions (Pagano, 1969, 1970a,b) when considering composite laminates, revealed that shear stress distributions are much more distorted than that. At the same time, numerous displacement approaches were suggested. The roughest suggestion for taking into account transverse shear strains, $\varepsilon_{\alpha 3}$, is assuming that $\varepsilon_{\alpha 3}$ is uniform through the thickness (First Order Shear Deformation Theory). Yet, it leads to too stiff shear behavior and necessitates the introduction of shear correction factors (Mindlin, 1951; Whitney, 1972). Above all, this assumption enforces a discontinuous shear stress $\sigma_{\alpha 3}$ through the thickness. Other models have been designed (Reddy, 1984; Touratier, 1991; Vidal and Polit, 2008) to remove the use of shear correction factors, but most of them did not lead to continuous $\sigma_{\alpha 3}$, as indicated by Reddy (1989). The most refined Equivalent Single Layers models, which finally led to continuous shear stress are zigzag models (Ambartsumian, 1969; Whitney, 1969; Carrera, 2003). However, these models are restricted to some specific configurations (symmetry of the plate and material constitutive equation) and involve higher-order partial derivative equations than the simple Reissner–Mindlin plate model.

The difficulties encountered with transverse stress fields instigated the consideration of enriched models: Layerwise models. In these models, all plate degrees of freedom are introduced in each layer of the laminate. Continuity conditions are enforced between layers. In this area, most of the improvements have been focused on refining the local displacement field. The reader can refer to Reddy (1989) and Carrera (2002) for detailed reviews. It should be noted that a static approach has also been considered for layerwise models. Based on the variational formulation from Pagano (1978), it treats each layer as a Reissner–Mindlin plate and enforces stress continuity conditions (Naciri et al., 1998; Diaz Diaz et al., 2001, 2007; Hadj-Ahmed et al., 2001; Caron et al., 2006; Dallot and Sab, 2008). Both stress and displacement approaches for Layerwise models lead to correct estimates of local 3D fields. However their main drawback is that they involve many more degrees of freedom (proportional to the number of layers) than Equivalent Single Layer models.

Based on Reissner (1945) paper, the purpose of this work is to suggest an Equivalent Single Layer higher-order plate theory which gives an accurate enough estimate of transverse shear stresses in the linear elasticity framework. For this, we are motivated by two observations. The first one is that Kirchhoff–Love strain fields have clearly been identified as good first-order approximation for slender plates thanks to asymptotic expansion approaches. Thus, it would be inconsistent to refine in-plane fields further without introducing correct estimation of transverse fields. The second one is that the 3D equilibrium plays a critical role in the estimation of transverse shear stress in all the existing approaches. For instance, Whitney (1972) introduced 3D equilibrium in order to compute shear correction factors and more recently, when benchmarking several plate models, Noor and Malik (2000) used the 3D equilibrium to estimate shear stresses. We show in this paper that revisiting the use of 3D equilibrium in order to derive transverse shear stress as Reissner (1945) did for homogeneous plates leads to a full Bending Gradient plate theory. The Reissner–Mindlin theory will appear as a special case of the new Bending-Gradient theory when the plate is homogeneous.

The paper is organized as follows. In Section 2 notations are introduced. In Section 3 we recall briefly the full 3D elastic problem for a clamped plate and, in Section 4, how it is possible to derive plate equilibrium equations without any assumption on microscopic fields and how Reissner derived his shear stress distribution. Then we demonstrate in Section 5.1.1 that applying Reissner's approach for deriving transverse shear stress to a composite laminate involves more static shear degrees of freedom (DOF) than the usual shear forces $\boldsymbol{Q}$. The mechanical meaning of these new DOF is presented and compatible fields are identified in Section 5.2. The constitutive equation for the Bending Gradient is derived in Section 5.3 which leads to the formulation of a complete plate theory. Finally, in Section 6, it is demonstrated that for the special case of homogeneous plates, the Reissner–Mindlin and the Bending-Gradient plate theory are identical. Thus a means to quantitatively compare both theories is provided and applied to conventional laminates.

2. Notations

Vectors and higher-order tensors are boldfaced and different typefaces are used for each order: vectors are slanted: $\boldsymbol{T}$, $\boldsymbol{u}$. Second order tensors are sans serif: M, e. Third order tensors are in typewriter style: $\mathtt{\Phi}$, $\mathtt{\Gamma}$. Fourth order tensors are in calligraphic style $\mathcal{D}$, $\mathcal{c}$. Sixth order tensors are double stroked $\mathbb{F}$, $\mathbb{W}$.

When dealing with plates, both 2-dimensional (2D) and 3D tensors are used. Thus, $\widetilde{\boldsymbol{T}}$ denotes a 3D vector and $\boldsymbol{T}$ denotes a 2D vector or the in-plane part of $\widetilde{\boldsymbol{T}}$. The same notation is used for higher-order tensors: $\tilde{\boldsymbol{\sigma}}$ is the 3D second-order stress tensor while $\boldsymbol{\sigma}$ is its in-plane part. When dealing with tensor components, the indexes specify the dimension: a_{ij} denotes the 3D tensor $\tilde{\mathsf{a}}$ with Latin index $i, j, k.. = 1,2,3$ and $\mathsf{a}_{\alpha\beta}$ denotes the 2D tensor a with Greek indexes α, $\beta, \gamma.. = 1,2$. $\widetilde{\mathcal{C}} = \mathcal{C}_{ijkl}$ is the fourth-order 3D elasticity stiffness tensor. $\widetilde{\mathcal{S}} = \mathcal{S}_{ijkl} = \widetilde{\mathcal{C}}^{-1}$ is the fourth-order 3D elasticity compliance tensor while $\mathcal{c} = \mathcal{c}_{\alpha\beta\gamma\delta}$ denotes the plane-stress elasticity tensor. $\mathcal{c}$ is not the in-plane part of $\widetilde{\mathcal{C}}$ but it is the inverse of the in-plane part of $\widetilde{\mathcal{S}} : \mathcal{c} = \mathcal{S}^{-1}$. The identity for in-plane elasticity is $\imath_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$, where $\delta_{\alpha\beta}$ is Kronecker symbol ($\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, $\delta_{\alpha\beta} = 0$ otherwise).

The transpose operation $^t\bullet$ is applied to any order tensors as follows: $(^tA)_{\alpha\beta\ldots\psi\omega} = A_{\omega\psi\ldots\beta\alpha}$.

Three contraction products are defined, the usual dot product ($\tilde{\boldsymbol{a}} \cdot \tilde{\boldsymbol{b}} = a_i b_i$), the double contraction product ($\tilde{\mathsf{a}} : \tilde{\mathsf{b}} = \mathsf{a}_{ij}\mathsf{b}_{ji}$) and a triple contraction product ($\mathtt{A} \therefore \mathtt{B} = \mathtt{A}_{\alpha\beta\gamma}\mathtt{B}_{\gamma\beta\alpha}$). Einstein's notation on repeated indexes is used in these definitions. It should be noticed that closest indexes are summed together in contraction products. Thus, $\mathtt{\Phi} \cdot \boldsymbol{n} = \mathtt{\Phi}_{\alpha\beta\gamma} n_\gamma$ is different from $\boldsymbol{n} \cdot \mathtt{\Phi} = n_\alpha \mathtt{\Phi}_{\alpha\beta\gamma}$. The reader might easily check that $\imath : \imath = \imath$, $\imath \therefore \imath = 3/2\delta$ and that $\imath \cdot \imath = \imath_{\alpha\beta\gamma\delta}\imath_{\delta\epsilon\zeta\eta}$ is a sixth-order tensor. We recall also that resp. $\boldsymbol{a} \cdot \boldsymbol{b}$, $^t\mathsf{a} : \mathsf{b}$ and $^t\mathtt{a} \therefore \mathtt{b}$ define inner products and associated norms on $\mathbb{R}^2, (\mathbb{R}^2)^2$ and $(\mathbb{R}^2)^3$, respectively.

The derivation operator $\tilde{\boldsymbol{\nabla}}$ is also formally represented as a vector: $\tilde{\mathsf{a}} \cdot \tilde{\boldsymbol{\nabla}} = \mathsf{a}_{ij}\nabla_j = \mathsf{a}_{ij,j}$ is the divergence and $\mathsf{a} \otimes \boldsymbol{\nabla} = \mathsf{a}_{\alpha\beta}\nabla_\gamma = \mathsf{a}_{\alpha\beta,\gamma}$ is the gradient. Here $\otimes$ is the dyadic product.

Finally, the integration through the thickness is noted $\langle\bullet\rangle : \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x_3)dx_3 = \langle f\rangle$.

3. The 3D model

We consider a linear elastic plate of thickness h occupying the 3D domain $\Omega = \omega \times]-h/2, h/2[$, where $\omega \subset \mathbb{R}^2$ is the mid-plane of the plate (Fig. 1). Cartesian coordinates (x_1,x_2,x_3) in the reference frame $(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3)$ are used. The constitutive material is assumed to be invariant with respect to translations in the (x_1,x_2) plane. Hence,