International Journal of Solids and Structures 47 (2010) 1698–1710

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Strain gradient elastic homogenization of bidimensional cellular media

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ARTICLE INFO

Article history:
Received 5 October 2009
Received in revised form 10 March 2010
Available online 16 March 2010

Keywords:
Homogenization
Symmetry classes
Strain gradient elasticity
Anisotropy

ABSTRACT

The present paper aims at introducing an homogenization scheme for the determination of strain gradient elastic coefficients. This scheme is based on a quadratic extension of homogeneous boundary condition (HBC). It allows computing strain elastic effective tensors. This easy-to-handle computational procedure will then be used to construct overall behaviors and to verify some theoretical predictions on strain gradient elasticity.

1. Introduction

Lightweight and innovative materials design is nowadays one of the most important challenge for material engineering, the goal is to reach high mechanical properties with low density materials. To achieve such contradictory objectives, scientific community focused on mesoscale structured cellular materials. Such material design requires to understand at the same time the relation between architecture and physical properties, and the explicit method to calculate those properties.

According to a geometrical definition of a RVE (Representative Volume Elementary) a classical way to obtain the overall behavior of the cellular material is to use homogenization theory. It is well known that classical homogenization theory relies on a broad scale separation between geometric pattern and mechanical fields. If the scale separation is not broad enough, the classical theory fails to predict the overall behavior. As shown by Boutin (1996) and Forest (1998), keeping a continuum description requires to consider a generalized continuum to model the substitution material.

Especially, when designing millimetric microstructural materials to be implemented in centimetric structures (e.g. hollow spheres stacking for acoustical absorber (Gasser, 2003)) strong scale separation is not granted. Then, second-order elastic effects have to be taken into account in the homogenization approach.

As the number of elastic constants in strain gradient theories dramatically increases with the order of tensors, a systematic way of identifying such coefficients is necessary. We proposed here such a method by combining recent results in extended homogenization methods and symmetry properties of higher-order stiffness tensors:

- Quadratic homogenization scheme through the use of quadratic boundary conditions (Gologanu et al., 1997; Forest, 1999).
- Extended Voigt notations for different symmetry classes of second gradient elasticity (Auffray et al., 2009).

Our main results are 3-folds:

1. First, it is shown that the circular cavity shape used in several higher-order homogenization schemes (Gologanu et al., 1997; Zybell et al., 2008) leads to a singular sixth-order elasticity tensor.
2. Second, it is shown numerically that the application of the quadratic homogenization scheme provides isotropic second-order effective properties for octagonal and pentagonal cell shapes, thus illustrating purely mathematical considerations of symmetry.
3. Finally, an example of chiral dependent behavior is given, illustrating an exotic property of strain gradient elasticity.

These results confirm that generalized homogenization schemes are powerful tools to estimate higher-order elastic properties. Furthermore some useful operators to translate higher-order moduli from the equivalent first strain gradient into the second gradient of displacement theories, according to Mindlin's formulation, are provided.

To reach those objectives, several facts on strain gradient elasticity will be recalled in Section 2. Some basic definitions and results about symmetry classes (Auffray et al., 2009) will be

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doi:10.1016/j.ijsolstr.2010.03.011

summed up. In a second time, attention will be focused on the homogenization scheme. Extension of the effective modulus approach was firstly proposed by Gologanu et al. (1997) then used by Forest and Sab (1998). This approach will be detailed and specified to our problem. All the needed operators will be defined. The last section will be devoted to numerical experiments on different geometrical patterns. It will be shown that isotropic homogenization proceed on a circular shape (as proposed in Gologanu et al. (1997) and Zybell et al. (2008)) leads to a degenerated isotropic tensor that should not be used in computational simulation. Another construction based on Hermann theorem consequences (Auffray, 2008) will be proposed.

2. Mindlin's strain gradient elasticity

2.1. Constitutive law

In classical elasticity theory stress at a material point is linked to strain through the classical elasticity tensor. This relation, usually known as Hooke law, is written in tensorial fashion:

$$\sigma_{(ij)} = C_{\underline{(ij)(lm)}}\varepsilon_{(lm)} \tag{1}$$

with $\sigma_{(ij)}$ the symmetrical-stress tensor, $\varepsilon_{(lm)}$ the strain tensor and $C_{\underline{(ij)(lm)}}$ the tensor describing material elastic properties. The notation $()$ stands for the minor symmetries whereas $\underline{\ldots}$ stands for the major ones.

Second-grade elasticity is a kinematic enhancement of classical elasticity taking into account the second gradient of displacement in the mechanical formulation. Such a generalization could be constructed in, at least, three different, but equivalent, ways (Mindlin and Eshel, 1968). In this paper interest will be focused on the two first formulations.

In *type I* formulation, the freedom extra degrees will simply be defined as the second gradient of displacement:

$$\underset{\simeq}{\kappa}^{\mathrm{I}} = \underline{\mathrm{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} \tag{2}$$

whereas in *type II* formulation the strain gradient will be considered:

$$\underset{\simeq}{\kappa}^{\mathrm{II}} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla} \tag{3}$$

These two definitions solely differ by the index symmetry of $\underset{\simeq}{\kappa}$. We have got $\kappa^{\mathrm{I}}_{i(jk)}$ and $\kappa^{\mathrm{II}}_{(ij)k}$ with the following relations between those two systems:

$$\kappa^{II}_{ijk} = \frac{1}{2}\left(\kappa^{I}_{ijk} + \kappa^{I}_{jik}\right) \tag{4}$$

$$\kappa^{I}_{ijk} = \kappa^{II}_{ijk} + \kappa^{II}_{kij} - \kappa^{II}_{jki} \tag{5}$$

As these two systems are defined up to a permutation, other properties will be introduced just for *type II* elasticity: strain gradient elasticity (SGE).

Taking into account strain gradient effect in the mechanical formulation leads to define symmetrically the hyperstress tensor $\tau_{(ij)k}$. In each material point, the knowledge of the stress and the hyperstress tensors allows to compute the effective tensor $\eta_{(ij)}$. This tensor is defined as:

$$\eta_{(ij)} = \sigma_{(ij)} - \tau_{(ij)k,k} \tag{6}$$

It is the tensor to be considered to calculate the local equilibrium (Forest, 2004). Tensors $\sigma_{(ij)}$ and $\tau_{(ij)k}$ are related with $\varepsilon_{(lm)}$ and $\kappa_{(lm)n}$ through the following general constitutive law:

$$\sigma_{(ij)} = C_{\underline{(ij)(lm)}}\varepsilon_{(lm)} + M_{(ij)(lm)n}\kappa_{(lm)n} \tag{7}$$

$$\tau_{(ij)k} = M_{(ij)k(lm)}\varepsilon_{(lm)} + A_{\underline{(ij)k(lm)n}}\kappa_{(lm)n} \tag{8}$$

where the tensor $A_{\underline{(ij)k(lm)n}}$ is the second-order elasticity tensor and $M_{(ij)(lm)n}$ the coupling tensor between first and second-order elasticity.

In a 3-D space this coupling tensor will vanish for a centro-symmetric media (Triantafyllidis and Bardenhagen, 1996). In 2-D space this tensor vanishes for any media that is even order rotational invariant (Auffray et al., 2008). For both cases the constitutive relation could be rewritten as follow:

$$\sigma_{(ij)} = C_{\underline{(ij)(lm)}}\varepsilon_{(lm)} \tag{9}$$

$$\tau_{(ij)k} = A_{\underline{(ij)k(lm)n}}\kappa_{(lm)n} \tag{10}$$

To switch constitutive law from one system to another, the following operators could easily be defined:

$$P^{\mathrm{I}\to\mathrm{II}}_{ijklmn} = \frac{1}{2}\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{kn} \tag{11}$$

$$P^{\mathrm{II}\to\mathrm{I}}_{ijklmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jl}\delta_{km} \tag{12}$$

where $\underset{\approx}{\mathrm{P}}^{\mathrm{I}\to\mathrm{II}}$ stands for the operator from *type I* to *type II*, and conversely. The following relation holds true:

$$P^{\mathrm{I}\to\mathrm{II}}_{ijkopq}P^{\mathrm{II}\to\mathrm{I}}_{opqlmn} = 1^{\mathrm{II}}_{ijklmn} \tag{13}$$

where $\underset{\approx}{1}^{\mathrm{II}}$ should not be confused with the sixth-order identity tensor; a symmetrical relation could also be defined for $\underset{\approx}{1}^{\mathrm{I}}$.

Switching between the two systems is related to the fact that most of our theoretical results are demonstrated in *type II* second-grade elasticity whereas the boundary conditions needed for the homogenization scheme are more natural in *type I* formulation. As transformations from one system to another are straightforward, it seems interesting to point out how to transfer results. Let us now introduce some results about SGE anisotropic tensors.

2.2. Anisotropic tensors

Most of the results presented here could be found and detailed in Auffray et al. (2009) and Auffray (2009). A true tensorial representation for SGE tensor will first be introduced. Then for each material symmetry group the corresponding physical group will be given including minimal number of coefficient of the associated tensor. For the hereafter studied anisotropic systems the corresponding Voigt representations will be given.

2.2.1. Voigt tensorial representation

In order to handle the second-order elastic tensor, a mathematical transformation could be introduced to turn the two-dimensional sixth-order tensor into a six-dimensional second-order tensor.[1] This transformation allows rewriting the second-order constitutive relation as[2]:

$$\hat{\tau}_{\alpha} = \widehat{A}_{(\alpha\beta)}\hat{\kappa}_{\beta} \tag{14}$$

A rigorous way of representing the sixth-order tensor *A* as a second-order one according to its symmetries is:

[1] The permutation order-dimension is just a coincidence, in 3-D the same transformation would turn a three-dimensional sixth-order tensor into a 18-dimensional second-order tensor.

[2] The hat notation ^ indicates a second-order representation of a sixth-order tensor.