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Random homogenization analysis in linear elasticity based on analytical bounds and estimates

Juan Ma [a,b,*], Ilker Temizer [b], Peter Wriggers [b]

[a] School of Electronical and Mechanical Engineering, Xidian University, Xi'an 710071, PR China
[b] Faculty of Mechanical Engineering, Leibniz Universität Hannover, 30167 Hannover, Germany

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ABSTRACT

In this work, random homogenization analysis of heterogeneous materials is addressed in the context of elasticity, where the randomness and correlation of components' properties are fully considered and random effective properties together with their correlation for the two-phase heterogeneous material are then sought. Based on the analytical results of homogenization in linear elasticity, when the randomness of bulk and shear moduli, the volume fraction of each constituent material and correlation among random variables are considered simultaneously, formulas of random mean values and mean square deviations of analytical bounds and estimates are derived from Random Factor Method. Results from the Random Factor Method and the Monte-Carlo Method are compared with each other through numerical examples, and impacts of randomness and correlation of random variables on the random homogenization results are inspected by two methods. Moreover, the correlation coefficients of random effective properties are obtained by the Monte-Carlo Method. The Random Factor Method is found to deliver rapid results with comparable accuracy to the Monte-Carlo approach.

1. Introduction

The homogenization method has been developed and extended to reduce the number of composite design parameters significantly by the introduction of effective characteristics using potential or complementary energy principles (Markovic and Ibrahimbegovic, 2006; Aboudi, 1991; Zohdi and Wriggers, 2005). The method relies on a statistically representative sample of material, referred to as a representative volume element (RVE). It is a finite sized sample from the heterogeneous material that characterizes its macroscopic behavior (Aboudi, 1991; Zohdi and Wriggers, 2005; Torquato, 2002). Although this technique, in its modern version, is more than 40 years old, there are many novel approaches and applications, such as in the food industry (Kanit, 2006), some composites made of wood (Lux, 2006), superconductors (Kaminski, 2005), even for time-dependent cases by "equation free" approach (Samaey et al., 2006); a variety of materially nonlinear multi-component composites can be homogenized as well (Idiart, 2006). Following numerous engineering applications, the strength of composites can also be estimated by the homogenization method (Steeves and Fleck, 2006).

Homogenization techniques deliver effective properties of heterogeneous materials. Exact computational approaches are summarized in Zohdi and Wriggers (2005). Here, the attention is focused to estimates and bounds. In this context, early approximations for the effective properties were first developed by Voigt (1889) and Reuss (1929). In 1957, Eshelby (1957) obtained a relatively compact solution that has been a basis for many approximation methods. Based on variational principles, Hashin and Shtrikman (1962) developed a model that improved solutions of the effective properties. Additional classical models have been proposed to estimate the effective properties, including the Self-Consistent method, the dilute distribution method, and the Mori and Tanaka (1973) method. Further approaches for estimating or bounding the effective responses of heterogeneous materials can be found for instance in Aboudi (1991), Mura (1987) and Nemat-Nasser and Hori (1999).

In recent years, a lot of attention is paid to random composites because of an uncertainty in reinforcement location/shape and/or pore spatial distribution in matrices, and randomness in components. Kaminski reported the perturbation-based homogenization analysis of two-phase composites (Kaminski and Kleiber, 2000) and the perturbation-based homogenization analysis for thermal conductivity of unidirectional fiber reinforced composites (Kaminski, 2001). Sakata obtained a macroscopic response by applying stochastic homogenization analysis for unidirectional fiber reinforced composites using the Monte-Carlo simulation (Sakata

* Corresponding author at: School of Electronical and Mechanical Engineering, Xidian University, Xi'an 710071, PR China.

E-mail addresses: juan@ikm.uni-hannover.de (J. Ma), temizer@ikm.uni-hannover.de (I. Temizer), wriggers@ikm.uni-hannover.de (P. Wriggers).

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et al., 2008). Sakata also reported the three-dimensional results of perturbation analysis for the homogenized elastic tensor and the equivalent elastic properties (Sakata et al., 2008) or the second-order perturbation-based homogenization method (Sakata et al., 2008). Kaminski also developed a higher order perturbation-based analysis (Kaminski, 2007). Ostoja-Starzewski (2002) and Xu and Brady (2005) designed other approaches like the Fourier Galerkin method for random homogenization analysis. So far, most of the analytical models still consider the randomness of the geometric configuration like shape, size, location and distribution of particles, and the perturbation-based homogenization analysis is used as a main solution. For composites with different constituents, randomness of physical properties and volume fraction of the different constituents has an important effect on the effective properties after homogenization. Especially, the correlation among random variables should be fully considered, as should be the correlation among final random results. The perturbation method is based on the hypothesis that a random variable has a small perturbation about the mean value and subsequently Taylor series is used to describe a random variable as the sum of a determinate part plus a perturbation part that together transform the nonlinear equations into linear recursion formulas. For this reason, it is easy to get the first-order perturbation expansion, but a great deal of computation is needed to get the second-order or higher-order perturbation terms and the final results because of second-order or higher-order partial derivatives included. Moreover, this approach can quickly become numerically intractable when a large number of random variables are involved (Kaminski and Kleiber, 2000). Finally, due to the existence of secular terms, the accuracy and application of perturbation method is limited to some degree.

The goal of this work is to solve the random homogenization problem by two different methods while completely considering the randomness and correlation of the heterogeneous material. Based on the summary of the analytical results regarding the estimation of effective linear elasticity parameters, random effective properties of the two-phase heterogeneous materials are analyzed by the Random Factor Method (RFM) and the Monte-Carlo Method (MCM), in which the randomness of the bulk and shear moduli, volume fractions of the two constituents and the correlation among the random variables are considered fully. The numerical characteristics of effective properties after homogenization are derived by means of the random variable's moment method, and they are then compared with those obtained by MCM in order to verify the effectiveness of the method given in this paper. A future aim along this direction is to introduce the uncertainty to the finite element analysis of linear and nonlinear heterogeneous materials and multiscale engineering problems with heterogeneities distributed over multiple length scales.

2. Random analysis of the analytical bounds and estimates for the effective elasticity moduli

2.1. Monte-Carlo Method (MCM)

Monte-Carlo Method, the alternative to RFM, is used to solve the random problem by the test of random samples. According to the principle of MCM, samples associated with every random variable should be generated from their probabilistic distribution and correlation. Each sample realization is analyzed to obtain target quantities that display a statistical distribution. For normal distributions that are typically obtained in homogenization techniques, this statistical distribution is characterized by the mean value and the mean square deviation. In most Monte-Carlo simulations, different random variables are assumed to be independent of each other. However, this assumption does not hold in many engineering problems. Based on the Cholesky factorization of covariance matrix of random vector (Ali Touran, 1992), the simulation of the correlation among random variables is realized by MCM in this work. Some computational results and conclusions about correlation of random variables are given in Section 3.3.

2.2. Random Factor Method

The main ideas of Random Factor Method (RFM) (Ma et al., 2006; Gao et al., 2004) are as follows. A random variable y can be expressed as a random factor $\tilde{y}$ multiplied by its mean value $\mu_y : y = \tilde{y} \cdot \mu_y$. The random factor represents the randomness of the variable; its mean value is 1.0 and its mean square deviation is that of the random variable. $\tilde{y}$ obeys the same probabilistic distribution as y.

The main analyzing procedure by RFM in this paper is: firstly, the constituent's random variable is expressed as its random factor multiplied by its mean value; secondly, the material's effective properties are then written as random factors of constituents' random properties and volume fractions multiplied by their mean values respectively, that is, the material's effective properties are the functions of these random factors; finally, the mean values and mean square deviations of the effective properties can be obtained by using moment method of random variables.

RFM can directly and clearly reflect the influence of any random variable on the results. Additionally, based on the random variables' moment method, it is easy to consider the effect of correlation among random variables on homogenization results by RFM.

In the following, aiming at two-phase heterogeneous materials, the analytical bounds (Reuss–Voigt bounds (RV), Hashin–Shtrikman bounds (HS)) and estimates (Maxwell/Mori–Tanaka model (MW), Self-Consistent model (SC), Differential model (DF)) listed in Tables 1 and 2 (Torquato, 2002) will be considered. The mean values and mean square deviations of random effective properties will be derived by RFM.

In Tables 1 and 2, the volume average $\langle(\cdot)\rangle \overset{def}{=} V^{(1)}(\cdot)^{(1)} + V^{(2)}(\cdot)^{(2)}, \langle(\tilde{\cdot})\rangle \overset{def}{=} V^{(1)}(\cdot)^{(2)} + V^{(2)}(\cdot)^{(1)}; k^{(1)}, u^{(1)}, V^{(1)}$ are the bulk and shear moduli and the volume fractions of the first type of constituent; $k^{(2)}$, $u^{(2)}$ and $V^{(2)}$ are those of the second type of constituent; $f(x,y) = \frac{y[dx/2+(d+1)(d-2)y/d]}{x+2y}, g(x) = \frac{2(d-1)}{d}x$; for HS model, it is assumed that $k^{(2)} \geqslant k^{(1)}$, $u^{(2)} \geqslant u^{(1)}$.

Considering the randomness of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$ and $V^{(2)}$ simultaneously, they can be written as $k^{(1)} = \tilde{k}^{(1)} \cdot \mu_{k^{(1)}}, k^{(2)} = \tilde{k}^{(2)} \cdot \mu_{k^{(2)}}, u^{(1)} = \tilde{u}^{(1)} \cdot \mu_{u^{(1)}}, u^{(2)} = \tilde{u}^{(2)} \cdot \mu_{u^{(2)}}$ and $V^{(2)} = \widetilde{V}^{(2)} \cdot \mu_{V^{(2)}}$ from RFM, where $\tilde{k}^{(1)}$, $\tilde{u}^{(1)}$, $\tilde{k}^{(2)}$, $\tilde{u}^{(2)}$, $\widetilde{V}^{(2)}$ are the random factors of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ respectively, mean values of random factors are 1.0 and their mean square deviation are those $\sigma_{k^{(1)}}, \sigma_{u^{(1)}}, \sigma_{k^{(2)}}, \sigma_{u^{(2)}}, \sigma_{V^{(2)}}$ of $k^{(1)}$, $u^{(1)}$, $k^{(2)}$, $u^{(2)}$, $V^{(2)}$ respectively, $\mu_{k^{(1)}}, \mu_{u^{(1)}}, \mu_{k^{(2)}}, \mu_{u^{(2)}}, \mu_{V^{(2)}}$ are mean values of every random variable respectively.

For RV, HS and MW models in Tables 2 and 1, they are explicit and can be generally described as follows:

$$k^* = G\left(\tilde{k}^{(1)}\mu_{k^{(1)}}, \tilde{u}^{(1)}\mu_{u^{(1)}}, \tilde{k}^{(2)}\mu_{k^{(2)}}, \tilde{u}^{(2)}\mu_{u^{(2)}}, \widetilde{V}^{(2)}\mu_{V^{(2)}}\right), \quad (1)$$

Table 1
A summary of analytical estimates for the effective elasticity moduli for spherical particles in d-dimensions.

Model	k^*	u^*
SC	$\sum_{i=1}^2 V^{(i)} \frac{k^{(i)}-k^*}{k^{(i)}+g(u^*)} = 0$	$\sum_{i=1}^2 V^{(i)} \frac{u^{(i)}-u^*}{u^{(i)}+f(k^*,u^*)} = 0$
DF	$\frac{dk^*}{dV^{(2)}} = \frac{1}{1-V^{(2)}}[k^* + g(u^*)]\frac{k^{(2)}-k^*}{k^{(2)}+g(u^*)}$	$\frac{du^*}{dV^{(2)}} = \frac{1}{1-V^{(2)}}[u^* + f(k^*,u^*)]\frac{u^{(2)}-u^*}{u^{(2)}+f(k^*,u^*)}$
MW	$\frac{k^*-k^{(1)}}{k^*+g(u^{(1)})} = V^{(2)}\left[\frac{k^{(2)}-k^{(1)}}{k^{(2)}+g(u^{(1)})}\right]$	$\frac{u^*-u^{(1)}}{u^*+f(k^{(1)},u^{(1)})} = V^{(2)}\left[\frac{u^{(2)}-u^{(1)}}{u^{(2)}+f(k^{(1)},u^{(1)})}\right]$

Here Maxwell/Mori–Tanaka model yields identical results to the Mori–Tanaka model, although the derivation procedures are slightly different (Torquato, 2002).