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Stability of long transversely-isotropic elastic cylindrical shells under bending

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A B S T R A C T

The present paper investigates buckling of cylindrical shells of transversely-isotropic elastic material subjected to bending, considering the nonlinear prebuckling ovalized configuration. A large-strain hypoelastic model is developed to simulate the anisotropic material behavior. The model is incorporated in a finite-element formulation that uses a special-purpose "tube element". For comparison purposes, a hyperelastic model is also employed. Using an eigenvalue analysis, bifurcation on the prebuckling ovalization path to a uniform wrinkling state is detected. Subsequently, the postbuckling equilibrium path is traced through a continuation arc-length algorithm. The effects of anisotropy on the bifurcation moment, the corresponding curvature and the critical wavelength are examined, for a wide range of radius-to-thickness ratio values. The calculated values of bifurcation moment and curvature are also compared with analytical predictions, based on a heuristic argument. Finally, numerical results for the imperfection sensitivity of bent cylinders are obtained, which show good comparison with previously reported asymptotic expressions.

1. Introduction

The application of bending loading in long cylinders causes cross-sectional distortion, resulting in loss of bending stiffness and a nonlinear moment–curvature equilibrium path. This phenomenon, referred to as "ovalization" or "Brazier effect", was first analyzed by Brazier (1927) for the case of elastic isotropic cylinders. Reissner (1959) also investigated ovalization response for both initially straight and curved tubes, taking under consideration the effects of pressure. In addition to cross-sectional ovalization, the increased axial stress at the compression zone of the ovalized cylinder causes bifurcation instability (buckling) in a form of longitudinal wavy-type "wrinkles". This often occurs before a limit moment is reached on the moment–curvature ovalization path and constitutes a buckling problem associated with a highly nonlinear ovalized prebuckling state, where the compression zone of the deformed cylindrical shell wall has a double curvature of opposite sign in the longitudinal and in the hoop direction.

An approximate, yet quite efficient method to determine the bifurcation point on the ovalized prebuckling moment–curvature path has been proposed by Axelrad (1965) for long isotropic elastic tubes, considering a rigorous ovalization analysis. Axelrad assumed that for a bent cylinder buckling would occur when the local compressive longitudinal stress σ_x at any point around the tube circumference reaches the critical stress σ_{cr} of an axially-compressed circular cylinder with radius equal to the local radius $r_{\theta 0}$ of the ovalized section (often referred to as "equivalent cylinder"). This buckling condition can be expressed as follows $\sigma_x = \sigma_{cr}$, and using classical Donnell shell buckling equation (Brush and Almroth, 1975), one readily obtains:

$$\sigma_x = \frac{E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{r_{\theta 0}}\right) \quad (1)$$

where $1/r_{\theta 0}$ is the deformed (ovalized) cross-sectional curvature at the center of the buckling zone, E is Young's modulus and ν is Poisson's ratio. In subsequent works (Stephens et al., 1975; Fabian, 1977; Ju and Kyriakides, 1992; Karamanos, 2002; Houliara and Karamanos, 2006), more elaborate solutions for determining the bifurcation point on the ovalization path of isotropic elastic cylinders have been reported, which verified the good accuracy of the simplified formula (1) proposed in Axelrad (1965). In addition, the corresponding buckling half wavelength of the isotropic cylinders was found to be very close to the value of the half wavelength for the axisymmetric buckling mode of the uniformly-compressed equivalent cylinder L_{hw}^{axi}, when radius-to-thickness ratio is increased $(r/t \to \infty)$.

$$L_{hw}^{axi} = \pi\sqrt{r_{\theta 0} t}\Big/\left[12\left(1-\nu^2\right)\right]^{1/4} \quad (2)$$

The initial post-bifurcation behavior of isotropic elastic cylinders under bending has been investigated in Fabian (1977), using an asymptotic analysis. It was demonstrated that the bifurcation mode is symmetric, followed by an unstable initial postbuckling behavior. Numerical results in Ju and Kyriakides (1992) verified the initially unstable postbuckling behavior of a bent elastic cylinder.

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Furthermore, the results from the more refined numerical formulation in Karamanos (2002) and Houliara and Karamanos (2006) also demonstrated the unstable nature of the initial postbuckling response, in the form of a "snap-back".

The increasing use of composite materials in engineering applications has motivated many studies in the elastic stability of anisotropic shells. Kedward (1978) enhanced Brazier's solution considering a different modulus for the longitudinal and the hoop direction, and obtained a closed-form expression for the ovalization and the moment–curvature path. Spence and Toh (1979) extended Reissner's formulation (Reissner, 1959) to account for orthotropic material behavior, using nonlinear finite deflection thin shell theory. These numerical results were compared with experimental test data from steel and "melenex" cylinders. Stockwell and Cooper (1992) presented a direct extension of Reissner's isotropic cylinder formulation (Reissner, 1959) to obtain a closed-form expression for the moment–curvature relationship, and the analytical results were compared with numerical results from a commercial finite element program. Libai and Bert (1994) used thin-shell theory and a mixed variational principle to investigate the nonlinear ovalization behavior of anisotropic cylinders, and reported solutions for long, medium-length and short cylinders, including the effects of end boundary conditions. Furthermore, a closed-form expression for the moment–curvature ovalization path was derived for infinitely long cylinders. Corona and Rodrigues (1995) presented numerical results for orthotropic cylinders with cross-ply layers. Using nonlinear ring theory, the prebuckling ovalization solution was determined. Bifurcation was detected through a perturbation technique, based on non-shallow cylindrical shell kinematics. Tatting et al. (1996) analyzed long anisotropic cylinders with symmetric lay-ups through a finite-difference solution of semi-membrane shell equations. The final moment–curvature expression was identical to the one reported by Reissner (1959) for isotropic tubes, and its solution was found to compare well with the simplified Kedward solution (Kedward, 1978). Using the Kedward solution for the prebuckling path, bifurcation was examined, considering the simplified engineering hypothesis of "equivalent cylinder" proposed in Axelrad (1965), as well as the effects of internal or external pressure, taking into account the influence of laminate stacking sequence. It was found that bifurcation occurs before a limit point is reached on the ovalization path. Harursampath and Hodges (1999) developed an enhanced beam model accounting for cross-sectional deformation, in terms of trigonometric series expansion, to analyze long, thin-walled cylinders of anisotropic materials. Employing one term of series expansion, they provided closed-form expressions for the ovalization path and the corresponding stresses. Using this simplified solution, limit-moment instability, local buckling and material ply failure were examined. Recently, Wadee et al. (2007) using an analytical formulation and a second-degree trigonometric series solution in the hoop direction, obtained results for anisotropic cylinders, in terms of the ultimate moment and the buckling wavelength.

The present paper investigates bifurcation instability and postbuckling behavior of long elastic cylinders under bending loading, using a numerical formulation and solution methodology, with special emphasis on material modeling. The material of the cylinder is considered to be transversely-isotropic, where the anisotropy axis is initially directed in the longitudinal cylinder direction. A hypoelastic model is developed for describing the mechanical behavior of transversely-isotropic materials, through a stress rate co-rotational with the anisotropy axis. Furthermore, a hyperelastic model based on a quadratic free energy function is also considered for comparison purposes. The material models are embedded in a special-purpose nonlinear finite element formulation, which accounts for the presence of pressure. The effects of anisotropy on the buckling moment, the corresponding curvature, the buckling wavelength, and the shape of the buckling modes are examined, considering a wide range of radius-to-thickness ratio values (r/t). Furthermore, using an "arc-length" continuation technique, postbuckling paths are traced for different levels of anisotropy, and the sensitivity of cylinder response in the presence of initial imperfections is examined. Finally, an analytical approach, based on the concept of "equivalent cylinder", is presented leading to closed-form expressions for the buckling moment, curvature and the corresponding wavelength, which are compared with the corresponding numerical results.

2. Numerical formulation and solution

The nonlinear formulation adopted in the present work was introduced in its general form by Needleman (1982). It has been employed for the nonlinear analysis of relatively thick elastic–plastic offshore tubular members (Karamanos and Tassoulas, 1996) and, more recently, for the elastic stability of thin-walled isotropic cylinders under bending and pressure (Karamanos, 2002; Houliara and Karamanos, 2006).

The cylinder is considered as an elastic continuum. A net of coordinate lines embedded in and deforming with the continuum is employed. The embedded coordinates are denoted by ξ^i $(i = 1, 2, 3)$, where coordinate lines ξ^1 and ξ^2 are directed in the hoop and in the axial direction of the cylinder, respectively, and for a constant value of ξ^3, they define a shell lamina, whereas the coordinate line ξ^3 is initially directed through the shell thickness. Therefore, $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, \xi^3, t)$ is the position vector of the material point (ξ^1, ξ^2, ξ^3) in the current (deformed) configuration at time t. The position of the material point (ξ^1, ξ^2, ξ^3) at $t = 0$ in the reference (undeformed) configuration is denoted by $X = X(\xi^1, \xi^2, \xi^3)$. At any material point, the covariant base vectors in the reference and in the current configuration, which are tangent to the coordinate lines, are denoted by $\mathbf{G}_i = \partial\mathbf{X}/\partial\xi^i = \mathbf{X}_{,i}$ and $\mathbf{g}_i = \partial\mathbf{x}/\partial\xi^i = \mathbf{x}_{,i}$, respectively, where the comma denotes partial differentiation with respect to ξ^i. Furthermore, $\mathbf{G}^k$ and $\mathbf{g}^k$ denote the contravariant (reciprocal) base vectors in the reference and current configuration, respectively.

2.1. Governing equations

Deformation is described by the rate-of-deformation (stretch) tensor $\mathbf{d}$, which is the symmetric part of the velocity gradient $\mathbf{L}$, whereas the anti-symmetric part of $\mathbf{L}$ is the continuum spin $\boldsymbol{\omega}$. It can be shown that the covariant components of the rate-of-deformation tensor are:

$$d_{kl} = \frac{1}{2}\left(V_{m/l}(\mathbf{G}^m \cdot \mathbf{g}_k) + V_{m/k}(\mathbf{G}^m \cdot \mathbf{g}_l)\right) \tag{3}$$

where $V_{m/l}$ is the covariant derivative of the velocity vector components with respect to the reference basis.

Equilibrium is expressed through the principle of virtual work, considering an admissible displacement field $\delta\mathbf{u}$. For a continuum occupying the region V_0 and V in the reference and in the current configuration, respectively, and with boundary B in the deformed configuration, the principle of virtual work is expressed as:

$$\int_{V_0} \delta U_{i/j}\left(\mathbf{G}^i \cdot \mathbf{g}_k\right)\tau^{kj}\, dV_0 = \int_B \delta\mathbf{u} \cdot \mathbf{t}\, dB_q + M\delta\theta \tag{4}$$

where $\mathbf{t}$ is the surface traction, M is the moment, $\delta\theta$ is the variation of end rotation, τ^{ij} are the contravariant components of the Kirchhoff stress tensor τ that is parallel to the Cauchy stress $\boldsymbol{\sigma}$ $(\tau dV_0 = \boldsymbol{\sigma} dV)$ and