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Two-dimensional Eshelby's problem for two imperfectly bonded piezoelectric half-planes

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ABSTRACT

General solutions are derived to the two-dimensional Eshelby's problem of an inclusion of arbitrary shape embedded in one of two imperfectly bonded anisotropic piezoelectric half-planes. The inclusion undergoes uniform eigenstrains and eigenelectric fields. In this work four different kinds of imperfect interface models with vanishing thickness are considered: (i) a compliant and weakly conducting interface, (ii) a stiff and highly conducting interface, (iii) a compliant and highly conducting interface, and (iv) a stiff and weakly conducting interface. Furthermore the obtained general solutions are illustrated in detail through an example of an elliptical inclusion near the imperfect interface. It is observed that the full-field expressions of the three analytic function vectors characterizing the electroelastic field in the two piezoelectric half-planes including the elliptical inclusion can be elegantly and concisely presented through the introduction of an integral function. We also present the tractions and normal electric displacement along a compliant and weakly conducting imperfect interface induced by the elliptical inclusion. It is found that the imperfection of the interface has no influence on the leading term in the far-field asymptotic expansion of the tractions and normal electric displacement along the compliant and weakly conducting interface induced by an arbitrary shaped inclusion. The far-field expansions of the analytic function vectors in the two imperfectly bonded half-planes for an arbitrary shaped inclusion are also derived. Some new identities and structures of the matrices $\mathbf{N}_i$ and $\mathbf{N}_i^{(-1)}$ for anisotropic piezoelectric materials are obtained.

1. Introduction

The Eshelby's problem of an inclusion with eigenstrains (or transformation strains) has been a topic in micromechanics for more than fifty years (Eshelby, 1957; Mura, 1987). When addressing the three-dimensional Eshelby's problem, the Green's function approach is prevalent (Eshelby, 1957; Mura, 1987; Nozaki and Taya, 2001). However when discussing two-dimensional (2D) Eshelby's problem in isotropic or anisotropic solids, the complex variable method is more effective (see for example Jaswon and Bhargava, 1961; Bhargava and Radhakrishna, 1964; Willis, 1964; Yang and Chou, 1976, 1977; Ru, 2000, 2001; Pan, 2004; Jiang and Pan, 2004; Wang et al., 2007). It has been found in recent years that studies on Eshelby's problem are essential in understanding the behaviors of quantum dots and quantum wires in nanocomposite solids (see recent reviews by Ovid'ko and Sheinerman, 2005 and Malanganti and Sharma, 2005).

When addressing the inclusion problems in a two-phase infinite medium (say with a flat interface), it is found that the perfect interface assumption was adopted in the majority of the previous studies (see for example, Zhang and Chou, 1985. Yu and Sanday, 1991; Jiang and Pan, 2004). In a recent study, Wang et al. (2007) considered a 2D thermal inclusion of arbitrary shape embedded in one of two imperfectly bonded isotropic elastic half-planes by using Muskhelishvili's complex variable method (Muskhelishvili, 1963). The imperfect interface in that study was simulated by using the linear spring layer with vanishing thickness. However, the corresponding Eshelby's problem for two imperfectly bonded dissimilar anisotropic piezoelectric half-planes still remains a challenging problem.

It is of interest to point out also that so far various interface models have been proposed to simulate an interphase layer with finite thickness (Needleman, 1990; Benveniste and Miloh, 2001; Benveniste and Baum, 2007; Bertoldi et al., 2007a,b; Benveniste, 2006, 2009), to account for damage (for example, micro-cracks and micro-voids) occurring on the interface (Fan and Sze, 2001), and to study their influence on the effective properties of the composites (Lu and Lin, 2003; Wang and Pan, 2007) and on the interfacial wave propagation (Melkumyan and Mai, 2008). Nondistructive evaluation methods were also proposed to detect and characterize the interface imperfection (Nagy, 1992; Hu and Nagy, 1998). It was reported that the effect of interfacial stress, defects, impurities,

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and electrodes on the variation of polarization in ferroelectric thin films could be significant (Lu and Cao, 2002). However, as expected that if the piezoelectricity of an interphase layer is taken into consideration (Benveniste, 2009), the scenarios of the imperfect interface will become more complex in view of the fact that now the interface has imperfection in both elasticity and dielectricity.

In this work we consider the 2D problem of an Eshelby inclusion of arbitrary shape with uniform eigenstrains and eigenelectric fields embedded in one of two bonded anisotropic piezoelectric half-planes by means of the Stroh formalism (Suo et al., 1992; Suo, 1993; Wang, 1994; Chung and Ting, 1996; Ru, 2000, 2001). In extending previous works (Ru, 2001; Pan, 2004; Jiang and Pan, 2004; Wang et al., 2008), the two anisotropic piezoelectric half-planes are now bonded through a thin anisotropic piezoelectric layer. It is found that closed-form solutions can be derived when the middle piezoelectric layer is replaced by an imperfect interface with vanishing thickness. The imperfect interface models discussed in this work can be classified into the following four different kinds:

(i) Compliant and weakly conducting interface. This imperfect interface is based on the assumption that tractions and normal electric displacement are continuous across the interface, whereas the elastic displacements and electric potential undergo jumps on the interface which are proportional to the interface tractions and normal electric displacement.
(ii) Stiff and highly conducting interface. This imperfect interface is based on the assumption that displacements and electric potential are continuous across the interface, whereas tractions and normal electric displacement undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and electric potential.
(iii) Compliant and highly conducting interface. This imperfect interface is based on the assumption that tractions and tangential electric field are continuous across the interface, whereas the elastic displacements and charge potential undergo jumps on the interface which are proportional to the interface tractions and tangential electric field.
(iv) Stiff and weakly conducting interface. This imperfect interface is based on the assumption that displacements and charge potential are continuous across the interface, whereas tractions and tangential electric field undergo jumps on the interface which are proportional to certain surface differential operators of the interface displacements and charge potential.

Our theoretical development demonstrates that the parameters in all the four kinds of imperfect interface models can be explicitly expressed in terms of the electroelastic moduli and the thickness of the piezoelectric layer.

2. The Stroh formalism for anisotropic piezoelectric materials

In the following we will present two different schemes of the Stroh formalism. Scheme 1 of the Stroh formalism will be adopted in the analyses of a compliant and weakly conducting interface (Section 3), and a stiff and highly conducting interface (Section 4). Scheme 2 will be adopted in the analyses of a compliant and highly conducting interface (Section 5), and a stiff and weakly conducting interface (Section 6).

2.1. Scheme 1 of the Stroh formalism

The basic equations for an anisotropic piezoelectric material can be expressed in a fixed rectangular coordinate system $x_i (i = 1, 2, 3)$ as

$$\begin{aligned}&\sigma_{ij} = C_{ijkl}u_{k,l} + e_{kij}\phi_{,k}, \quad D_k = e_{kij}u_{i,j} - \epsilon_{kl}\phi_{,l},\\ &\sigma_{ij,j} = 0, \quad D_{i,i} = 0,\end{aligned} \tag{1}$$

where repeated indices mean summation, a comma follows by i $(i = 1, 2, 3)$ stands for the derivative with respect to the ith spatial coordinate; u_i and ϕ are the elastic displacement and electric potential; σ_{ij} and D_i are the stress and electric displacement; C_{ijkl}, ϵ_{ij} and e_{ijk} are the elastic, dielectric and piezoelectric coefficients, respectively.

For 2D problems in which all quantities depend only on x_1 and x_2, the general solutions can be expressed as (Suo et al., 1992; Wang, 1994; Ting, 1996)

$$\begin{aligned}&\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \phi]^T = \mathbf{Af}(z) + \overline{\mathbf{Af}(z)},\\ &\mathbf{\Phi} = [\Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \varphi]^T = \mathbf{Bf}(z) + \overline{\mathbf{Bf}(z)},\end{aligned} \tag{2}$$

where

$$\begin{aligned}&\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4],\\ &\mathbf{f}(z) = [f_1(z_1) \quad f_2(z_2) \quad f_3(z_3) \quad f_4(z_4)]^T,\\ &z_i = x_1 + p_i x_2, \quad \mathrm{Im}\{p_i\} > 0, \quad (i = 1-4),\end{aligned} \tag{3}$$

with

$$\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^T\end{bmatrix}\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix} = p_i\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix}, \quad (i = 1-4) \tag{4}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}, \tag{5}$$

and

$$\mathbf{Q} = \begin{bmatrix}\mathbf{Q}^E & \mathbf{e}_{11}\\ \mathbf{e}_{11}^T & -\epsilon_{11}\end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix}\mathbf{R}^E & \mathbf{e}_{21}\\ \mathbf{e}_{12}^T & -\epsilon_{12}\end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix}\mathbf{T}^E & \mathbf{e}_{22}\\ \mathbf{e}_{22}^T & -\epsilon_{22}\end{bmatrix}, \tag{6}$$

$$(\mathbf{Q}^E)_{ik} = C_{i1k1}, \quad (\mathbf{R}^E)_{ik} = C_{i1k2}, \quad (\mathbf{T}^E)_{ik} = C_{i2k2}, \quad (\mathbf{e}_{ij})_m = e_{ijm}. \tag{7}$$

In addition the extended stress function vector $\mathbf{\Phi}$ is defined, in terms of the stresses and electric displacements, as follows:

$$\begin{aligned}&\sigma_{i1} = -\Phi_{i,2}, \quad \sigma_{i2} = \Phi_{i,1}, \quad (i = 1-3)\\ &D_1 = -\varphi_{,2}, \quad D_2 = \varphi_{,1}.\end{aligned} \tag{8}$$

Here we can call φ a charge potential (Suo, 1993). Due to the fact that the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following normalized orthogonal relationship:

$$\begin{bmatrix}\mathbf{B}^T & \mathbf{A}^T\\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T\end{bmatrix}\begin{bmatrix}\mathbf{A} & \overline{\mathbf{A}}\\ \mathbf{B} & \overline{\mathbf{B}}\end{bmatrix} = \mathbf{I}, \tag{9}$$

then three real Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ can be introduced

$$\mathbf{S} = \mathrm{i}(2\mathbf{AB}^T - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{AA}^T, \quad \mathbf{L} = -2\mathrm{i}\mathbf{BB}^T. \tag{10}$$

During this investigation, the following identities will also be utilized:

$$\begin{aligned}&2\mathbf{A}\langle p_\alpha\rangle\mathbf{A}^T = \mathbf{N}_2 - \mathrm{i}(\mathbf{N}_1\mathbf{H} + \mathbf{N}_2\mathbf{S}^T),\\ &2\mathbf{A}\langle p_\alpha\rangle\mathbf{B}^T = \mathbf{N}_1 + \mathrm{i}(\mathbf{N}_2\mathbf{L} - \mathbf{N}_1\mathbf{S}),\\ &2\mathbf{B}\langle p_\alpha\rangle\mathbf{B}^T = \mathbf{N}_3 + \mathrm{i}(\mathbf{N}_1^T\mathbf{L} - \mathbf{N}_3\mathbf{S}),\end{aligned} \tag{11}$$

where $\langle * \rangle$ is a 4×4 diagonal matrix in which each component is varied according to the Greek index α (from 1 to 4).

It can also be easily checked that

$$\begin{bmatrix}\mathbf{N}_1^{(-1)} & \mathbf{N}_2^{(-1)}\\ \mathbf{N}_3^{(-1)} & \mathbf{N}_1^{(-1)^T}\end{bmatrix}\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix} = \frac{1}{p_i}\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix}, \quad (i = 1-4) \tag{12}$$

where

$$\mathbf{N}_1^{(-1)} = -\mathbf{Q}^{-1}\mathbf{R}, \quad \mathbf{N}_2^{(-1)} = -\mathbf{Q}^{-1}, \quad \mathbf{N}_3^{(-1)} = \mathbf{T} - \mathbf{R}^T\mathbf{Q}^{-1}\mathbf{R}. \tag{13}$$

The detailed structures and identities of $\mathbf{N}_i$ and $\mathbf{N}_i^{(-1)}$ $(i = 1, 2, 3)$ for Scheme 1 can be found in Appendix A.