International Journal of Solids and Structures 47 (2010) 365–373

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Limit behavior of Koiter model for long cylindrical shells and Vlassov model

F. Béchet [a], O. Millet [b,*], É. Sanchez-Palencia [c]

[a] Laboratoire de Mécanique de Lille, Université de Lille 1, France
[b] Laboratoire d'Etude des Phénomènes de Transfert Appliqués aux Bâtiments, Université de La Rochelle, France
[c] Institut Jean Le Rond d'Alembert, Université de Jussieu, France

ARTICLE INFO

Article history:
Received 5 March 2009
Received in revised form 17 September 2009
Available online 1 October 2009

Keywords:
Koiter model
Shell theory
Thin-walled beam
Vlassov model
Asymptotic expansion

ABSTRACT

We propose in this article to consider the limit behavior of the Koiter shell model when one of the characteristic length of the middle surface becomes very large with respect to the other. To do this, we perform a dimensional analysis of Koiter formulation which involves dimensionless numbers characterizing the geometry and the loading. Once reduced to a one-scale problem corresponding to thin-walled beams (long cylindrical shell), using asymptotic expansion technique, we address the limit behavior of Koiter model when the aspect ratio of the shell tends to zero. We prove that at the leading order, Koiter shell model degenerates to a one dimensional thin-walled beam model corresponding to the Vlassov one. Moreover, we obtain a general analytical expression of the geometric constants involved, that improves the empirical expression given by Vlassov.

1. Introduction

Thin structures (plates, shells, beams and thin-walled beams) are widely used in industries because they provide a maximum of stiffness with a minimum of weight. Among thin-structures, we classically distinguish plates (with zero curvature), shells (shallow and curved shells), beams and thin-walled beams. Thin-walled beams are at the cross of shells and classical beams (with full cross-section): they can be seen equivalently as beams whose profile of a cross-section is thin, or as very long shells.

Classical one-dimensional or two-dimensional models of plates, shells, beams and thin-walled beams were obtained historically using *a priori* kinematics and statics assumptions in the three-dimensional equilibrium equations (Koiter, 1960; Novozhilov, 1959; Vlassov, 1962). More recently, asymptotic approaches[1] enabled to justify rigorously most of these classical models (Ciarlet and Destuynder, 1979; Destuynder, 1985; Sanchez-Palencia, 1989a,b, 1990; Hamdouni and Millet, 2003a,b; Rigolot, 1977; Marigo et al., 1998. However, in spite of all these works, the Koitershell model was never justified by asymptotic approach[2] although it is one of the most used for computing linear elastic shell problems.

A general feature of these asymptotic approaches (according to boundary conditions), is that in the asymptotic behavior of the variational displacement approach, penalty terms naturally emerge, leading to a limit problem in a constrained sub-space. Generally, the sub-space so defined by the penalty terms, corresponds to the searched classical kinematics (Kirchhoff-Love in plate theory, Novozhilov–Donnell for shallow shells, or pure bendings for geometrically non-rigid shells).

The reference model for thin-walled beams in linear elasticity is the Vlassov model[3] (Vlassov, 1962), historically established using *a priori* kinematics and statics assumptions. Similar a priori assumptions are used in recent works dealing with thin walled beam theory and applications (Kim and Kim (2005), Bottoni et al. (2005), El Fatmi (2007)).

In the literature, there exists only a few works on the rigorous justification of Vlassov model using asymptotic methods. First results were obtained in (Rodriguez and Viaño, 1995, Rodriguez and Viaño, 1997, Trabucho and Viaño, 1996), using an approach based on an expansion at the second order with respect to the diameter of the beam, to obtain an enriched model. Then in a second time, the thickness is assumed to tend to zero: that leads to a thin-walled beam model similar to Vlassov one.However, it is well-known that these two operations do not commute and the result depends on the choice made.[4]

Afterwards, a justification of Vlassov model from the asymptotic expansion of the three-dimensional elasticity equations was

* Corresponding author.
E-mail address: olivier.millet@univ-lr.fr (O. Millet).

[1] Note that surfacic approaches also exist but are not considered in this paper. They generally need also some a priori assumption for the constitutive law of the structure considered (Valid, 1995).

[2] Indeed, the Koiter model contains both membrane and bending effects coupled at different order of magnitude.

[3] Which is a one-dimensional model composed of four differential equations.

[4] This is a classical result in homogenization of periodic structures (Caillerie, 1984, Lewinski, 1991).

0020-7683/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2009.09.024

proposed in (Grillet et al., 2000, Hamdouni and Millet, in press). In this work, the width and the thickness of the profile tend together to zero. This way, for thin-walled beams with open strongly bent cross-section, the asymptotic model obtained differs slightly from Vlassov one: a supplementary term coupling bending and twist remains in the bending reduced equations. In the same manner, an asymptotic thin-walled beam model was obtained for shallow profiles (Grillet et al., 2005), and an extension to the non-linear case was proposed in (Grillet et al., 2004). In (Volovoi and Hodges, 2000), the authors proposed an anisotropic thin-walled beam model obtained from the Koiter model[5] using the variational-asymptotic method. Finally, let us also notice the works of Diaz and Sanchez-Palencia (2007), where convergence results to a thin-walled beam model for shallow profiles was established starting from Novozhilov–Donnell model.

In this paper, we address the limit behavior of Koiter shell model when one of the characteristic dimension of the middle surface becomes much larger than the other. We prove in particular that the Koiter model degenerates to a one dimensional thin-walled beam model corresponding to Vlassov model, when the aspect ratio tends to zero. Moreover, we obtain a general analytical expression of the geometric constants involved, that improves the empirical expression given by (Vlassov, 1962).

The paper is organized as follows. Section 2 is devoted to the formulation of the problem and of the linear Koiter shell model for a cylindrical shell with open cross-section. Then in Sections 3 and 4, we perform a dimensional analysis of Koiter formulation that makes naturally appear dimensional numbers characterizing the geometry and the loading. Once reduced to a one-scale problem corresponding to thin-walled beams (or to long cylindrical shells), using asymptotic expansion technique, we examine the limit behavior of the variational Koiter model when the aspect ratio of the shell tends to zero.[6] We shall see that several penalty terms appears in the weak formulation of the Koiter model, leading to the classical Vlassov kinematics for thin-walled beams. Thus, without any a priori assumption, the leading term of the asymptotic expansion of the displacements is proved to verify the classical Vlassov kinematics. Then, in Section 5, we prove that the constrained Koiter problem posed in this sub-space of Vlassov kinematics, degenerates to four ordinary differential equations (in the variable of the mean fiber of the beam) characterizing the three components of the displacement, and the twist angle. The limit one-dimensional beam model so obtained for very long shells in then compared to Vlassov equilibrium equations, and to the results obtained in (Grillet, 2003; Hamdouni and Millet, in press) from the asymptotic expansion of the three-dimensional equations of linear elasticity.

2. Description of the problem

2.1. Geometry and parametrization

In all that follows, the dimensional variables will be denoted with a tilde, whereas the dimensionless one will be denoted without. Moreover, the over-lined functions will denote functions which depend only on the variable $\tilde{y}_1$, associated to the longitudinal direction of the generators of the cylindrical shell (Fig. 1).

Let us consider a linear elastic thin-walled beam with open cross-section, or equivalently a long cylindrical shell (Fig. 1). It is described by the open cylindrical middle surface $\widetilde{S}$ and the constant thickness $2h$. In this paper, we limit our study to thin-walled beams with open strongly curved profile, whose curvature is denoted $\tilde{c}$ and length d. The length of the beam is denoted L and is assumed to be much longer than the length d of the profile.

Let us consider a cartesian coordinate system (O, e_1, e_2, e_3) associated with the three-dimensional space, and a local coordinate system $(\tilde{p}, a_1, a_2, N)$ associated with each point p of the profile, where $a_1 = e_1$ is the direction of the generators (see Fig. 1). In the case of cylindrical shells, we can always consider a mapping $\widetilde{\Psi}(\tilde{y}_1, \tilde{y}_2)$ of the middle surface $\widetilde{S}$ such as (a_1, a_2, N) is orthonormal, where $(\tilde{y}_1, \tilde{y}_2) \in \widetilde{\Omega}$ denotes the local variables associated with the parametrization. Consequently, we have $a_{\alpha\beta} = \delta_{\alpha\beta}$, and the contravariant basis (a^1, a^2, N) confuses with the covariant basis (a_1, a_2, N). Therefore, in the next, we will use indifferently the covariant or the contravariant components of the considered vector and tensor fields. Vectors will be denoted $\tilde{u} = \widetilde{U}_i e_i$ in the cartesian basis and $\tilde{u} = \tilde{u}_i a_i$ in the local basis, with $a_3 = N$. Moreover, the coordinates of a point $\tilde{p} = \widetilde{\Psi}(\tilde{y}_1, \tilde{y}_2)$ will be denoted $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ in the cartesian basis.

We assume that the shell considered is subjected to forces with surface densities $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$, and is fixed or clamped at its two extremities $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ corresponding respectively to $\tilde{y}_1 = 0$ and $\tilde{y}_1 = L$. Moreover, the lateral boundary $\widetilde{\Gamma}_l$ is free.

2.2. The Koiter shell model

We consider in this paper long cylindrical shells in linear elasticity, whose mechanical behavior is described by the Koiter shell model. Its variational formulation for a shell with a thickness $2h$ subjected to a surface loading $\tilde{f}$ classically writes (Destuynder, 1985; Bernadou, 1996; Sanchez-Hubert and Sanchez-Palencia, 1997; Béchet et al., 2008):

Find $\tilde{u} \in \widetilde{V} = \{H^1(\tilde{\Omega}) \times H^1(\tilde{\Omega}) \times H^2(\tilde{\Omega})\}$, $\tilde{v}$ satisfying the kinematics boundary conditions, such that:

$$2h \int_{\widetilde{S}} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\lambda\mu}(\tilde{u}) \tilde{\gamma}_{\alpha\beta}(\tilde{v}) \, d\widetilde{S} + \frac{2h^3}{3} \int_{\widetilde{S}} A^{\alpha\beta\lambda\mu} \tilde{\rho}_{\lambda\mu}(\tilde{u}) \tilde{\rho}_{\alpha\beta}(\tilde{v}) d\widetilde{S} = \int_{\widetilde{S}} \tilde{f}^i \tilde{v}_i d\widetilde{S} \quad \forall \tilde{v} \in \widetilde{V} \tag{1}$$

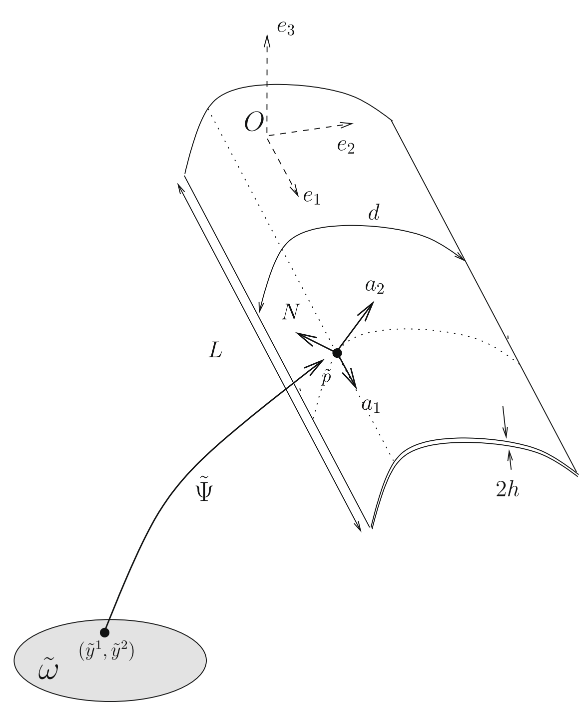

Fig. 1. The thin-walled beam and the used coordinate systems.

[5] Note that a different expression from (5) for the tensor of curvature variation is used.

[6] Equivalently when one the two lengths of the middle surface tends to infinity.