International Journal of Solids and Structures 46 (2009) 3927–3937

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Analytical solutions for the thick-walled cylinder problem modeled with an isotropic elastic second gradient constitutive equation

F. Collin [a], D. Caillerie [b], R. Chambon [b,*]

[a] FNRS Research Associate, ULg, Dpt Argenco, Institut de Mécanique et Génie, Civil Bât B52/3, Chemin des Chevreuils 1, B-4000 Liége 1, Belgium
[b] Grenoble Université Joseph-Fourier, Laboratoire 3S-R, INP, CNRS, B.P. 53X, 38041 Grenoble Cedex, France

ARTICLE INFO

Article history:
Received 11 June 2008
Received in revised form 27 February 2009
Available online 6 June 2009

Keywords:
Continuum mechanics
Micromorphic models
Second gradient models
Elasticity
Thick-walled cylinder
Analytical solutions

ABSTRACT

Numerical modeling of localization phenomena shows that constitutive equations with internal length scale are necessary to properly model the post-localization behavior. Moreover, these models allow an accurate description of the scale effects observed in some phenomena like micro-indentation. This paper proposes some analytical results concerning a boundary value problem in a medium with microstructure. In addition to their own usefulness, such analytical solutions can be used in benchmark exercises for the validation of numerical codes. The paper focuses on the thick-walled cylinder problem, using a general small strain isotropic elastic second gradient model. The most general isotropic elastic model involving seven different constants is used and the expression of the analytical solutions is explicitly given. The influence of the microstructure is controlled by the internal length scale parameter. The classical macrostress is no more in equilibrium with the classical forces at the boundary. Double stresses are indeed also generated by the classical boundary conditions and, as far as the microstructure effects become predominant (i.e. the internal length scale is much larger than the thickness of the cylinder), the macrostresses become negligible. This leads to solutions completely different from classical elastic ones.

1. Introduction

Since many years, the interest for enhanced models is increasing more and more. Many reasons explain this renewal. The advanced analysis of localization phenomena has shown that constitutive equations with internal length are necessary to properly model the experimental results involving some localized patterns (see, for instance, the pioneering works of Aifantis (1984), Bazant et al. (1984), de Borst and Muhlhaus (1992), Vardoulakis and Sulem (1995)). Moreover, these models allow to properly describe scale effects observed in some phenomena like micro-indentation as described in Nix and Gao (1997), Sulem and Cerrolaza (2002) and Rashid et al. (2004), or more generally in micro- and nano-mechanics as detailed in Fleck et al. (1994) and Fleck and Hutchinson (1997).

Many enhanced models have been proposed in the literature, especially within the framework of plastic or damage theories. In this paper, we focus first on general theories not closely related to specific behaviors. These theories based on an enhancement of the kinematic itself can be traced back to the pioneering works of Toupin (1962), Mindlin (1964) and Germain (1973b). The starting point of this paper are the materials with microstructure, as defined by Mindlin (1964) and Germain (1973b). On the contrary to the theories involving a gradient of internal variables, only valid for some specific behavior modeling, the latter are able to generate several kinds of constitutive equations like elasticity (Mindlin, 1965) elastoplasticity (Fleck and Hutchinson, 1997; Chambon et al., 1996, 1998), viscoplasticity (Forest and Sievert, 2006) and also hypoplasticity and continuum damage (Chambon et al., 1998). Adding some mathematical constraints to the most general materials with microstructure yields a large set of models. Among all these models, the first and most famous one is the Cosserat model (see Cosserat and Cosserat, 1909). Some of these models have been extensively studied and it has been demonstrated that general plastic (see Chambon et al., 2001) or viscoplastic models (see Forest and Sievert, 2006) can be derived within this framework. Even large elastoplasticity with multiplicative decomposition of the deformation gradient can be developed at least in the particular case of second gradient model (see Chambon et al., 2004). These models have been used in numerical codes, especially once more for the particular case of second gradient model (see, for instance, Chambon et al. (1998) in the one-dimensional case or Shu et al. (1999) for the elastic two-dimensional case and Matsushima et al. (2002) for the elastoplastic two-dimensional case).

However, as far as we know, there are very few analytical results concerning boundary value problems involving media with

* Corresponding author.
E-mail addresses: F.Collin@ulg.ac.be (F. Collin), Denis.Caillerie@hmg.inpg.fr (D. Caillerie), rene.chambon@hmg.inpg.fr (R. Chambon).

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doi:10.1016/j.ijsolstr.2009.05.017

microstructure. This is a problem because, in addition to their own usefulness, such solutions can be used as benchmark exercises in order to assess the validation of numerical codes. In one-dimensional cases, analytical solutions for a second gradient medium are provided in Chambon et al. (1998, 2001). In true bi-dimensional cases, Eshel and Rosenfeld (1970) and Bleustein (1966) propose the analytical solution of the stress concentration, respectively, at a cylindrical hole in a field of uniaxial tension and at a spherical cavity in a field of isotropic tension. Eshel and Rosenfeld (1975) have also developed the general equations of axi-symmetric problems in second gradient elastic materials, without giving explicit analytical solutions. Seeking the solution to the plastic expansion prior to the fully plastic stage, Zhao et al. (2007) provide a semi-analytical solution for a thick-walled cylinder. In this paper, the analytical solutions of the thick-walled cylinder problem in the case of a general small strain isotropic elastic second gradient model are given. With respect to the work of Zhao et al. (2007), it has to be emphasized that there is no restriction as far as the model is concerned. The more general isotropic elastic model with seven constants is used. Moreover, in the present paper, the analytical solutions are given explicitly, with the constants of integration depending on the prescribed boundary conditions. This allows to generate all the solutions of this problem by solving a set of four algebraic equations in four unknowns.

The sequence of the paper is as follows. The Section 2 is a presentation of the notations. Enhanced models require the use of unusual tensor. Moreover, the problem studied here has to be written in cylindrical coordinate system. In order to avoid any confusion, this section devoted to notation is necessary. The Section 3 is a presentation of the media with microstructure and relatives. This allows us to detail the link between the general media with microstructure and the second gradient model coming from the previous one by adding a mathematical constraint on the kinematic description. We follow mainly the works of Germain (see e.g. Germain (1973a,b)). The names of the models used in this paper is our choice. This is necessary since there is no general agreement concerning this point. In Section 4, the equations to be solved in the case of the thick-walled cylinder are derived for a second gradient model. The method uses extensively the virtual work principle. Both balance equations and boundary conditions are obtained. The fifth part deals with the resolution of the ordinary differential equation obtained in the previous section. The general method for the determination of the integration constants (from the prescribed boundary conditions) is presented. In a sixth part, some particular solutions corresponding to different boundary conditions are exhibited. The influence of the internal length scales introduced either by the model or by the boundary conditions are also exemplified. Some concluding remarks end up this paper.

2. Notations

Since all along this paper only orthonormal basis are used, it is not necessary to distinguish between covariant and contravariant components. We use then lower case subscripts to denote components of vectors and tensors. The other indices, namely superscripts, have other meanings and cannot be confused with the power operation because this later operation has a specific notation (see the first item in the following list). Vectors are denoted with arrows. The summation convention with tensorial indices is used. Let us emphasize that, using cylindrical coordinates, summation is meaningless for indices r, θ and z, even if they are in lower position.

- $(\alpha)^n$ means α to the power n.
- δ_{ij} are the components of the identity tensor (i.e. the Kronecker symbol).
- x_j are the components of the coordinates with respect to an orthonormal Cartesian basis.
- n_j are the components of the unit outward normal of a bounded domain.
- u_i are the components of the displacement field $\vec{u}$.
- $\partial_i a$ or $\partial_i(a)$ in some cases to avoid some ambiguities denotes the partial derivative of any quantity a with respect to the coordinate i.
- ϵ_{ij} are the components of the gradient of the displacement field i.e. $\epsilon_{ij} = \partial_j u_i$.
- ε_{ijk} are the components of the second gradient of the displacement field i.e. $\varepsilon_{ijk} = \partial_k(\partial_j u_i)$.
- χ_{ijk} are the components of the double stress denoted χ.
- σ_{ij} are the components of the macrostress denoted $\sigma, \sigma_{ij} = \sigma_{ji}$.
- τ_{ij} are the components is the microstress denoted τ.
- Ω is a given regular bounded domain.
- $\partial\Omega$ is the boundary of Ω assumed to enjoy the $C1$-continuity property.
- $*$ denotes virtual kinematical quantities.
- . denotes the scalar product of two vectors, for instance, $\vec{a}.\vec{b} = a_i b_i$.
- : denotes the scalar product of two second order tensors, for instance, $A : B = A_{ij}B_{ij}$.
- $\therefore$ denotes the scalar product of two third order tensors, for instance, $C \therefore D = C_{ijk}D_{ijk}$.
- ∇ is the gradient operator, which applies either to scalar or vector fields.
- $D(q)$ is the normal derivative of the quantity q, being a scalar or a vector. This means that: $D(q) = \partial_j(q)n_j = \nabla q.\vec{n}$.
- $D_j(q)$ are the components of the tangential derivative of the quantity q, being a scalar or a vector. $D_j(q) = \partial_j(q) - D(q) n_j = (\nabla q - (\nabla q.\vec{n})\vec{n})_j$.
- $\otimes$ denotes the dyadic product.
- $\vec{e}$ denotes generically a basis vector. The basis vectors of the orthonormal cartesian coordinates are denoted generically $\vec{e}_i$. The basis vectors of the cylindrical coordinates are denoted: $\vec{e}_r, \vec{e}_\theta$ and $\vec{e}_z$. This means that for any vector $\vec{a}$ we have $\vec{a} = a_i\vec{e}_i = a_r\vec{e}_r + a_\theta\vec{e}_\theta + a_z\vec{e}_z$. It may be worth reminding that the vectors $\vec{e}_r, \vec{e}_\theta$ depend on θ and that $\partial_\theta\vec{e}_r = \vec{e}_\theta$ and $\partial_\theta\vec{e}_\theta = -\vec{e}_r$.
- R^i and R^e denote, respectively, the inner and outer radii of the studied thick-walled cylinder.
- For a scalar function u depending only on r, the function v is defined such that: $v = \partial_r u + \frac{u}{r} = \frac{1}{r}\partial_r(ru)$.

To make these notations more clear, let us give some examples. Using orthonormal cartesian coordinates, we have, for instance, for a vector: $\nabla\vec{a} = \partial_j\vec{a} \otimes \vec{e}_j = \partial_j(a_i)\vec{e}_i \otimes \vec{e}_j$ and for a second order tensor $\nabla A = \partial_k A \otimes \vec{e}_k = \partial_k A_{ij}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$.

Using cylindrical coordinates yields the following well-known results, useful in the following:

$$\nabla\vec{a} = \partial_r\vec{a} \otimes \vec{e}_r + \frac{1}{r}\partial_\theta\vec{a} \otimes \vec{e}_\theta + \partial_z\vec{a} \otimes \vec{e}_z \quad (1)$$

and

$$\nabla A = \partial_r A \otimes \vec{e}_r + \frac{1}{r}\partial_\theta A \otimes \vec{e}_\theta + \partial_z A \otimes \vec{e}_z. \quad (2)$$

3. Media with microstructure and some relatives

3.1. Media with microstructure

The kinematic of a classical continuum is defined by a displacement field denoted $\vec{u}$, function of the coordinates. For media with microstructure, a field of a (not necessarily symmetric) second order tensor denoted E is added in the kinematic description.