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On the singularities of a constrained (incompressible-like) tensegrity-cytoskeleton model under equitriaxial loading

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ABSTRACT

Singularity theory is applied for the study of the characteristic three-dimensional tensegrity-cytoskeleton model after adopting an incompressibility constraint. The model comprises six elastic bars interconnected with 24 elastic string members. Previous studies have already been performed on non-constrained systems; however, the present one allows for general non-symmetric equilibrium configurations. Critical conditions for branching of the equilibrium are derived and post-critical behaviour is discussed. Classification of the simple and compound singularities of the total potential energy function is effected. The theory is implemented into the cusp catastrophe for the case of one-dimensional branching of the buckling-allowed tensegrity model, and an elliptic umbilic singularity for compound branching of a rigid-bar model. It is pointed out that singularity studies with constraints demand a quite different mathematical approach than those without constraints.

1. Introduction

The concept of tensegrity was initially apprehended and materialized in the sculptures of the artist Kenneth Snelson in 1948 and later patented by the architect R. Buckminster Fuller as a new method for designing geodesic structures (Fuller, 1961). Tensegrities are reticulated structures forming a highly geometric combination of bars and strings in space. In fact, tensegrity is a portmanteau word for "tension-integrity" referring to the integrity of structures as being based in a synergy between balanced continuous tension (elastic strings) and discontinuous compression (elastic bars) components. Pre-existing tensile stress in the string members, termed prestress, is required even before the application of any external loading in order to maintain structural stability. There already exists an extensive literature regarding the mechanics and advanced mathematics used for the integral description of these structures (e.g., Roth and Whiteley, 1981; Motro, 1992; Connelly and Back, 1998; Skelton et al., 2002; Lazopoulos, 2005a; Pirentis and Lazopoulos, 2006; Williams, 2007). The principal characteristics of tensegrity architecture are: the fact that elastic bars bear compressive load whereas elastic strings bear tensional load; prestress provides structural stability (self-equilibrated system); structural rigidity is proportionate to prestress; and the manifestation of "action at a distance" (Stamenović, 2006).

Almost three decades ago, the hypothesis that the *cytoskeleton* (CSK) is organised according to the principles of tensegrity architecture was introduced (Ingber et al., 1981; Ingber and Jamieson, 1985). The CSK is the intracellular filamentous biopolymer network whose constant remodelling directly affects almost all functions of living cells (Suresh, 2007). In the course of several years an ever increasing number of experimental observations have established that adherent cell behaviour is controlled and determined by its physical deformation and, especially, the deformation of the CSK. In the cellular tensegrity model and in terms of cell physiology, elastic bars correspond to microtubules while elastic strings correspond to the actin and intermediate filaments network (Stamenović, 2006, and references therein for an excellent overview). It has been found that the aforementioned mechanical properties of tensegrity systems are characteristic of the CSK as well (Ingber, 1993, 1998, 2008; Volokh et al., 2000; Stamenović, 2006). Actually, some of these properties were initially predicted by the tensegrity model and were later verified in laboratory experiments as mechanical properties of the CSK (Ingber, 2008).

In a series of previous theoretical studies by the authors, several aspects and properties of tensegrity architecture and CSK modelling have been discussed (Lazopoulos, 2005a,b; Pirentis and Lazopoulos, 2006; Lazopoulos and Lazopoulou, 2006a,b). The particular tensegrity model of this study has been used time and again during the last two decades for the investigation of the application of tensegrity architecture in CSK mechanics (Ingber, 1993; Stamenović et al., 1996; Coughlin and Stamenović, 1997, 1998; Wendling et al., 1999; Volokh et al., 2000; Wang and Stamenović, 2000).

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Appropriate modifications of the model have been presented in the literature in order to address specific problems from a broad range of phenomena related to the CSK (Stamenović, 2006; Ingber, 2008). However, previous studies by the authors and others did not take explicitly into account the fact that cells are almost incompressible. Here, motivated by this observation and in order to mimic cellular behaviour we integrated to the model a condition (constraint) that prohibits volumetric change, albeit, allows deformation.

In its initial unloaded configuration the tensegrity model under examination presents three-dimensional symmetry. Under the application of external equitriaxial loading and when the critical value is reached, the structure loses symmetry similarly to the case of Rivlin's cube (Rivlin, 1948, 1974; Ogden, 1997; Golubitsky et al., 1988). Two kinds of instabilities appear: the first is related to the global (overall) instability of the model and the second is due to local Euler buckling of the bars; both will be discussed later in the text along with the emergence of subsequent compound instabilities. The current tensegrity model, although simple, is a system with multiple degrees of freedom and presents a rich mechanical response that cannot be described in its entirety by force equilibrium concepts alone. To this end, an integral study of the model behaviour is effected by employing Singularity Theory for constrained systems – emerging from differential topology (Porteous, 1971, 1994). Even though formal branching theory demands such difficult tasks as the elimination of passive coordinates and the normalization of the total potential energy function (Thompson and Hunt, 1973; Vainberg and Trenogin, 1974; Troger and Steindl, 1991), the free coordinate bifurcation procedure elaborated here does not set these requirements. Applying the presented theory (Lazopoulos, 1994; Lazopoulos and Markatis, 1994), the various singularities of the total potential energy function exhibiting simple or compound branching of its equilibrium paths, under the influence of any constraints, can be investigated. In this formulation, explicit formulae of the critical conditions for branching are naturally derived and the classification of singularities provides the number and stability study of the post-critical equilibrium paths. The current treatment is an extension of previous work on the subject by including systems with constraints.

2. The tensegrity CSK model and its total potential energy function

The system under investigation comprises six inextensible elastic bars interconnected with a total of 24 elastic strings in the fashion shown in Fig. 1. Although the bars acquire an inextensible elastic curve, they may be buckled. In absence of external loading the initial geometrical configuration is dominated by prestress; specifically, the tensile forces carried by the strings are balanced by compression in the bars (Stamenović and Coughlin, 1999; Lazopoulos and Lazopoulou, 2006b).

In the initial configuration the self-equilibrated structure is fully symmetric and the origin of the Cartesian coordinate system is located in its centre. In fact, the three axes are aligned to the directions of the three pairs of parallel bars while the length of all string segments is the same. Following simple geometrical considerations it is straightforward to show that both the length of the string segments and the distance between the parallel bars are proportionate to the length of the bars (or, respectively, their chord-length if the bars are already buckled) (Kenner, 1976; Coughlin and Stamenović, 1997). The structure is subjected to a general three-dimensional loading such that forces of magnitude $T_X/2,\ T_Y/2,\ T_Z/2$ are applied at the endpoints of the bars (AA) and $(A'A')$, (BB) and $(B'B')$, (CC) and $(C'C')$, respectively (cf. Fig. 1). Thus, the parallel bars in each pair are either pulled apart or pushed towards each other, depending on whether the forces are

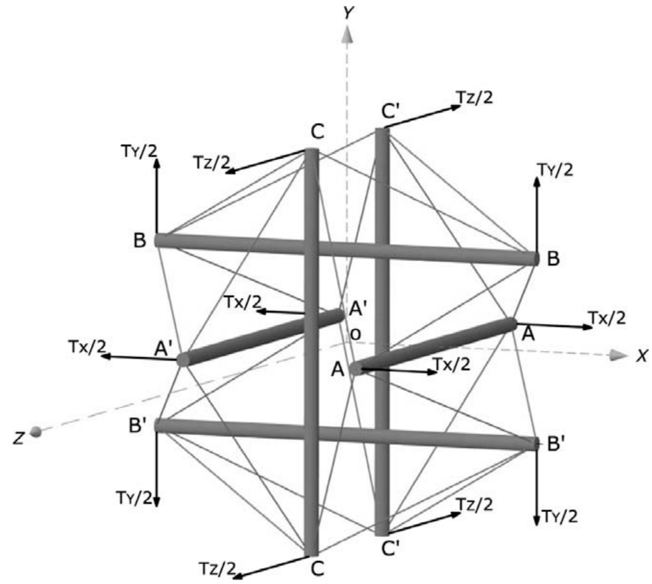

Fig. 1. Symmetric configuration of the tensegrity-cytoskeleton model.

extensive or compressive with respect to the overall system. Evidently, this results to a concerted change in all length values (current configuration). In the current configuration we call: $s_X,\ s_Y,\ s_Z$ the distances between the parallel bars (or chord-lengths) (AA) and $(A'A')$, (BB) and $(B'B')$, (CC) and $(C'C')$, respectively; l_1 the length of the string segments (AB), $(A'B)$, (AB'), $(A'B')$; l_2 the length of the string segments (AC), $(A'C)$, (AC'), $(A'C')$; and l_3 the length of the string segments (BC), $(B'C)$, (BC'), $(B'C')$. Then, simple geometrical arguments yield (Coughlin and Stamenović, 1997):

$$l_1 = \frac{1}{2}\sqrt{(L_2 - s_X)^2 + s_Y^2 + L_1^2}, \tag{1}$$

$$l_2 = \frac{1}{2}\sqrt{(L_1 - s_Z)^2 + s_X^2 + L_3^2}, \tag{2}$$

$$l_3 = \frac{1}{2}\sqrt{(L_3 - s_Y)^2 + s_Z^2 + L_2^2}, \tag{3}$$

where L_1, L_2 and L_3 are the current chord-lengths corresponding to the bars (AA) and $(A'A')$, (BB) and $(B'B')$, (CC) and $(C'C')$, respectively. If the deflection of the given buckled bar is defined by:

$$w_i(s) = \zeta_i \cdot \sin\left(\frac{\pi \cdot s}{L_0}\right), \quad i = 1,2,3, \tag{4}$$

where s and L_0 are the arc-length and length of the inextensible bar, respectively, and $\zeta_i \ll 1$, then the chord-length is expressed as:

$$L_i = L_0 - \frac{1}{2}\int_0^{L_0}\left(\left(\frac{dw_i(s)}{ds}\right)^2 - \frac{1}{4}\left(\frac{dw_i(s)}{ds}\right)^4\right)ds \simeq L_0 - \frac{\pi^2 \cdot \zeta_i^2}{4L_0}, \quad i = 1,2,3. \tag{5}$$

Now, if ρ denotes the curvature radius, the non-linear curvature is approximated as:

$$\left(\frac{1}{\rho}\right)_i \simeq \frac{d^2 w_i(s)}{ds^2}\left(1 + \frac{1}{2}\left(\frac{dw_i(s)}{ds}\right)^2\right), \quad i = 1,2,3. \tag{6}$$

Hence, the strain energy of each buckled bar is expressed as:

$$Wb_i = \frac{1}{2}k_B \int_0^{L_0}\left(\frac{1}{\rho}\right)_i^2 ds \simeq \frac{k_B \cdot \pi^4 \cdot \zeta_i^2}{4L_0^3}, \quad i = 1,2,3, \tag{7}$$