International Journal of Solids and Structures 47 (2010) 801–815

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

The generalized advancing conformal contact problem with friction, pin loads and remote loading – Case of rigid pin

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ARTICLE INFO

Article history:
Received 18 August 2009
Received in revised form 3 November 2009
Available online 3 December 2009

Keywords:
Contact
Singular integrals
Friction
Integral equation
Singular integro-differential equation
Conforming contact
Cradling
Rivet
Pin-loaded connection

ABSTRACT

The advancing, frictional contact problem for a rigid pin indenting an infinite plate with a circular hole is considered. The formulation is general, and considers remotely applied plate-stresses in addition to pin loads. Using the theory of generalized functions, it is found that the governing equation in full sliding is a singular integro-differential equation (SIDE). Partial-slip behavior is governed by an implicit, coupled singular integral equation (SIE) pair. Numerical solutions are presented for both types of problems. It is found that the contact tractions in monotonic loading become independent of the coefficient of friction above a certain threshold value. Finally, problems involving typical 'fretting-type' pin loads with and without remote-stresses are also investigated, revealing remarkable effects of the degree of conformality and load path on the steady-state traction distributions.

1. Introduction

Conformal contacts are important for a wide range of engineering applications, like fasteners. The distinguishing feature of this class of contacts is that the contacting bodies may not be treated as half-spaces because the contact size is significant when compared to the radius of curvature. Classically, two regimes have been identified for such contacts: 'cradling' and 'inclusion with remote-loading' which roughly correspond to conformal advancing and conformal receding contacts respectively. A significant body of literature exists in the frictionless case for both regimes, with a wide variety of methods being used for formulation. These include (1) Persson's direct formulation[1] using potentials for the disk and plate; (2) Dual-series techniques (Noble and Hussain, 1969) (3) Complex analytic stress-continuation (with contributions from Wilson, England and others) and (4) The recent method of To et al. (2007) which leads to approximate solutions, but may be used even when the plate is finite. The reader is referred to Gladwell (1980) and Ciavarella et al. (2006) for a more comprehensive list of references for frictionless problems.

By contrast, the literature for conformal contact problems with friction is somewhat sparse. Ho and Chau (1997) determined stress concentrations when body forces act on a (neat-fit) rivet loading an infinite plane. Iyer (2001) performed an FEM analysis for the case with finite plates and friction and discussed the applicability of various closed-form approximations. Finally, Hou and Hills (2001) obtained numerical solutions to the very special problem in which the pin and hole are elastically similar and almost conforming, using a distributed dislocation approach. In this instance, the gap between the two bodies was treated as an arc-shaped discontinuity in an otherwise continuous plate.

The present work treats advancing conformal contact problems with friction under general loading conditions i.e. pin loads and remote stresses applied in any path. The formulation is closed in the sense that it does not use infinite series, makes no assumptions regarding the tractions by way of symmetry or interactions and models the problem as exactly as possible within the limits of plane elasticity. In addition, a rapid numerical solution scheme for partial slip problems is also discussed.

While generalizing the problem in the directions mentioned, it is a useful simplification to regard the disk as rigid[2]; the contact pair is, nonetheless, a dissimilar one and may be considered a

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[1] His widely cited Ph.D. thesis is rather hard to obtain; fortunately, M. Ciavarella's summary in Ciavarella and Decuzzi (2001) is very useful and was adequate for the purpose of this work.

[2] It has been pointed out to us that this is a good approximation in applications since the pin/disk is usually less compliant than the plate/lug.

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doi:10.1016/j.ijsolstr.2009.11.019

prototypical case of advancing conformal contacts with friction. The present work also provides a useful template to proceed in case this simplification no longer holds.

2. Displacement fields caused by circumferential point loads

As a starting point, one begins with the complex potentials for a point load applied to the edge of a circular hole of radius a in an infinite elastic plane, with vanishing stresses and rotation at infinity. These were first derived by Rothman (1950) but he did not publish the displacement fields, which are surprisingly hard[3] to come by, so a brief derivation is presented here. The algebra is simplified somewhat by considering the normal and tangential load cases separately. Further, the load may be considered to act on a circumferential point located at an angle $\xi = 0$ and the result easily generalized for other angles. For a normal load N (positive outward) acting at an angle $\xi = 0$, the potentials are[4]

$$\Omega(z) = \frac{N}{2\pi}\left(-\log(z-a) + \frac{\kappa \log(z)}{(\kappa+1)}\right) \tag{1}$$

$$\omega(z) = \frac{N}{2\pi}\left(z\log(z-a) - \frac{z\log(z)}{\kappa+1} - a\log(z) + \frac{\kappa a^2}{(\kappa+1)z}\right) \tag{2}$$

The displacements in polar coordinates are given by

$$2G(u_r + iu_\theta) = e^{-i\theta}\left\{\kappa\Omega(z) - z\overline{\Omega'}(\bar{z}) - \overline{\omega'}(\bar{z})\right\} \tag{3}$$

Differentiating and substituting the potentials in Eq. (3) gives

$$\begin{aligned} 2G(u_r + iu_\theta) = & \frac{Ne^{-i\theta}}{2\pi}\left[-\kappa\log(z-a) + \frac{\kappa^2}{\kappa+1}z + \frac{z-\bar{z}}{\bar{z}-\bar{a}} - \frac{\kappa}{\kappa+1}\frac{z}{\bar{z}}\right. \\ & \left. - \log(\bar{z}-\bar{a}) + \frac{\log(\bar{z})}{\kappa+1} + \frac{a}{\bar{z}} + \frac{\kappa a^2}{\kappa+1}\frac{1}{\bar{z}^2}\right] \end{aligned} \tag{4}$$

On the surface, $z\bar{z} = a^2,\ z = ae^{i\theta}$ and $\bar{z} = ae^{-i\theta}$ so that, after simplifying and ignoring rigid body terms the *surface* displacements (indicated with a tilde) are given by

$$2G(\tilde{u}_r + i\tilde{u}_\theta) = \frac{Ne^{-i\theta}}{2\pi}\left[-\frac{\kappa+1}{2}\log(2-2\cos(\theta)) + (\kappa-1)i[A_1 - A_2]\right] \tag{5}$$

where

$$A_1 - A_2 = Arg\left(\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right)\right) \tag{6}$$

It is seen that $A_1 - A_2$ is a periodic linear function with discontinuities at $0, \pm 2m\pi$; it may be replaced by its values in a single period (suitably selected) with no loss of generality. Separating real and imaginary components gives the following surface displacements

$$2G\tilde{u}_r = \frac{N}{2\pi}\left[(\kappa-1)\sin(\theta)[A_1 - A_2] - \frac{\kappa+1}{2}\cos(\theta)\log(2-2\cos(\theta))\right] \tag{7}$$

$$2G\tilde{u}_\theta = \frac{N}{2\pi}\left[\frac{\kappa+1}{2}\sin(\theta)\log(2-2\cos(\theta)) + (\kappa-1)\cos(\theta)[A_1 - A_2]\right] \tag{8}$$

When the force acts at a point on the circumference situated at an angle ξ rather than 0,

$$2G\tilde{u}_r^N = \frac{N}{2\pi}\left[(\kappa-1)\sin(\theta-\xi)A(\theta,\xi) - \frac{\kappa+1}{2}\cos(\theta-\xi)L(\theta,\xi)\right] \tag{9}$$

$$2G\tilde{u}_\theta^N = \frac{N}{2\pi}\left[\frac{\kappa+1}{2}\sin(\theta-\xi)L(\theta,\xi) + (\kappa-1)\cos(\theta-\xi)A(\theta,\xi)\right] \tag{10}$$

[3] In sharp contrast to the ubiquitous Flamant solution

[4] Note that Rothman's $\omega'(z)$ represents what one would consider $\omega(z)$ in the canonical Kolosov–Muskhelishvili formulation

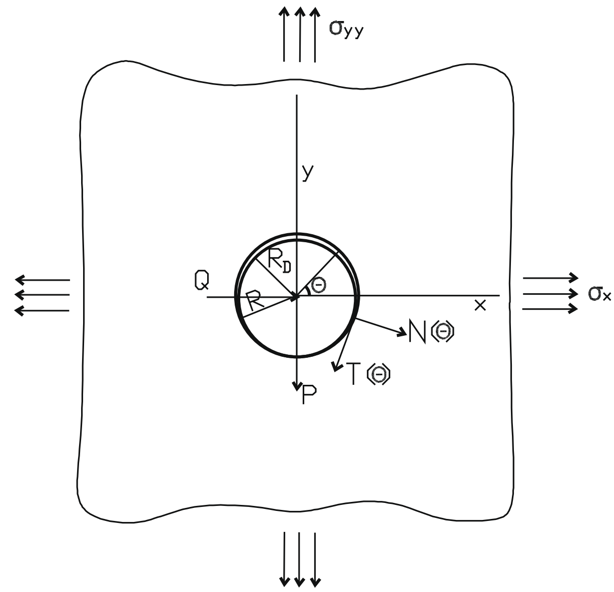

Fig. 1. Conformal contact.

where the following notation is used[5]

$$A(\theta,\xi) \equiv A_1 - A_2 = \frac{\theta-\xi}{2} - \frac{\pi}{2}sgn(\theta-\xi) \tag{11}$$

$$L(\theta,\xi) \equiv \log(2-2\cos(\theta-\xi)) \tag{12}$$

It is possible to proceed analogously for the case of a tangential point load T (clockwise positive) acting on a point on the circumference, in which case

$$2G\tilde{u}_r^T = \frac{T}{2\pi}\left[(\kappa-1)\cos(\theta-\xi)A(\theta,\xi) + \frac{\kappa+1}{2}\sin(\theta-\xi)L(\theta,\xi)\right] \tag{13}$$

$$2G\tilde{u}_\theta^T = \frac{T}{2\pi}\left[\frac{\kappa+1}{2}\cos(\theta-\xi)L(\theta,\xi) - (\kappa-1)\sin(\theta-\xi)A(\theta,\xi)\right] \tag{14}$$

3. Derivation of governing equations - full sliding

The full sliding *advancing* conformal contact problem may be interpreted (for example) as the moment-induced sliding of a disk (or shaft) of a radius slightly smaller than that of the hole under the applied pin-loads and remote stresses. In this case, the moment/torque at which the onset of sliding occurs is an unknown and must be determined as part of the solution.

Let the disk have radius R_D and the hole radius R, where $R_D < R$. In the undeformed configuration, the disk and hole are as shown in Fig. 1. In this state, the (first-order correct) gap function, $h_0(\theta)$, is

$$h_0(\theta) = (R - R_D)(1 + \sin(\theta)) \tag{15}$$

If the disk is now rotated by a small amount, and pressed into the elastic space by a rigid-body displacement vector $\overline{V}$ with horizontal component C_{0x} and vertical component Δ (positive downward), the new gap function, $h_d(\theta)$ is

$$h_d(\theta) = (R - R_D)(1 + \sin(\theta)) - C_{0x}\cos(\theta) + \Delta\sin(\theta) \tag{16}$$

Inside the contact, the overclosures thus produced should be relieved by elastic displacements so that the gap function, $h(\theta) = 0$. Now the gap function anywhere is defined as

$$h(\theta) = h_d(\theta) + \tilde{u}_r \tag{17}$$

so that inside the contact

[5] Again, the domain of the function $A(\theta,\xi)$ must be suitably selected.