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A new Kirchhoff plate model based on a modified couple stress theory

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ABSTRACT

In this paper a new Kirchhoff plate model is developed for the static analysis of isotropic micro-plates with arbitrary shape based on a modified couple stress theory containing only one material length scale parameter which can capture the size effect. The proposed model is capable of handling plates with complex geometries and boundary conditions. From a detailed variational procedure the governing equilibrium equation of the micro-plate and the most general boundary conditions are derived, in terms of the deflection, using the principle of minimum potential energy. The resulting boundary value problem is of the fourth order (instead of existing gradient theories which is of the sixth order) and it is solved using the Method of Fundamental Solutions (MFS) which is a boundary-type meshless method. Several plates of various shapes, aspect and Poisson's ratios are analyzed to illustrate the applicability of the developed micro-plate model and to reveal the differences between the current model and the classical plate model. Moreover, useful conclusions are drawn from the micron-scale response of this new Kirchhoff plate model.

1. Introduction

The behavior of micron-scale structures has been proven experimentally to be size dependent. Therefore, the classical continuum theory is inadequate to predict their response and the utilization of strain gradient (higher order) theories containing internal material length scale parameters is inevitable. For a literature review of the afore-mentioned theories can be found in the recent works of Vardoulakis and Sulem (1995); Exadaktylos and Vardoulakis (2001) and Tsepoura et al. (2002). Although, the strain gradient theories encounter the physical problem in its generality, they contain additional constants – besides the Lamé constants - which are difficult to determine even in their simplified form containing only two additional constants (Lam et al., 2003). Thus, gradient elasticity theories of only one additional material constant have been developed.

Altan and Aifantis (1992) suggested a simplified strain gradient model with only one strain gradient coefficient of length squared dimension which has been used by many investigators (e.g., Askes and Aifantis, 2002; Lazopoulos, 2004; Papargyri-Beskou and Beskos, 2008). A variational formulation of this simplified gradient elasticity theory has been presented by Gao and Park (2007) determining simultaneously both the equilibrium equations and the complete boundary conditions for the first time.

Yang et al. (2002) – modifying the classical couple stress theory (e.g. Mindlin, 1964; Koiter, 1964) – proposed a modified couple stress model in which only one material length parameter is needed to capture the size effect. This simplified couple stress theory is based on an additional equilibrium relation which force the couple stress tensor to be symmetric. So far has been developed for the static bending (Park and Gao, 2006) and free vibration (Kong et al., 2008) problems of a Bernoulli-Euler beam and for the static bending and free vibration problems of a Timoshenko beam (Ma et al., 2008). Moreover, Park and Gao (2008) solved analytically a simple shear problem after the derivation of the boundary conditions and the displacement form of the theory.

The work that has been done on the analysis of micro-plates is limited only to publications of linear and nonlinear plate models based on the simplified strain gradient model with one internal parameter introduced by Altan and Aifantis (1992). More specifically, Lazopoulos (2004) developed a strain gradient geometrically nonlinear plate model modifying Von Karman's nonlinear equations. This model was implemented in the study of the localized buckling of a long plate under uniaxial in-plane compression and small lateral loading, using the multiple scales perturbation method. Papargyri-Beskou and Beskos (2008) derived explicitly the governing equation of motion of gradient elastic flexural Kirchhoff plates, including the effect of in-plane constant forces on bending. In their work three boundary value problems were investigated (using the double Fourier series solution) dealing with static, stability and dynamic analysis of a rectangular simply supported gradient elastic flexural plate. However, the main drawback of the above plate models is that the presence of the microstructural effect raises the order of the resulting partial differential equation form four (classical case) to six (gradient case). As well as the classical boundary conditions are supplemented by additional (non-classical) ones containing higher order traction and higher

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order moments. Hence, the employed analytical solutions are restricted only to simple geometric shapes.

In this paper a new Kirchhoff plate model is developed for the static analysis of isotropic micro-plates with arbitrary shape based on the simplified couple stress theory of Yang et al. (2002) containing only one material length scale parameter which can capture the size effect. The proposed model is capable of handling plates with complex geometries and boundary conditions. To the author knowledge publications on the solution of the particular problem have not been reported in literature. The rest of paper is organized as follows. In Section 2 the total potential energy and its first variation of a three-dimensional body in rectangular coordinates are presented according to the modified couple stress theory. Using the minimum potential energy principle the governing equilibrium equation together with the pertinent boundary conditions in terms of the deflection are derived in their most general form, including elastic support or restraint, in Section 3. The resulting boundary value problem of the micro-plate is of the fourth order and it is solved using the Method of Fundamental Solutions (MFS) in Section 4. Several plates of various shapes, aspect and Poisson's ratios are analyzed in Section 5 to illustrate the developed micro-plate model and to reveal the differences between the current model and the classical plate model. Finally, a summary of conclusions is given in Section 6.

2. Modified coupled stress theory

In the modified couple stress theory of Yang et al. (2002), the strain energy density in rectangular coordinates of a three-dimensional body occupying a volume V bounded by the surface Ω is given as

$$U = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})dV \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \tag{3}$$

are the strain tensor and the symmetric part of the curvature tensor, respectively, u_i is the displacement vector and θ_i is the rotation vector defined as (Yang et al., 2002)

$$\theta_i = \frac{1}{2}e_{ijk}u_{k,j} \tag{4}$$

where e_{ijk} is the permutation symbol. In what it follows, unless otherwise stated, the Greek indices take the values 1, 2, while the Latin indices take the values 1, 2, 3. Moreover, σ_{ij} is the stress tensor and m_{ij} is the deviatoric part of the couple stress tensor given as

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{5}$$

$$m_{ij} = 2\mu l^2\chi_{ij} \tag{6}$$

where, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta and l is a material length scale parameter. From Eq. (3) it can be noted that the curvature tensor χ_{ij} is symmetric and consequently from Eq. (6) the couple stress tensor m_{ij} is also symmetric. That is, only the symmetric part of displacement gradient and the symmetric part of rotation gradient contribute to the deformation energy (Yang et al., 2002) which is different from that in the classical couple stress theory (e.g. Mindlin, 1964; Koiter, 1964).

Following Yang et al. (2002) and Park and Gao (2008), the work produced by the external forces is

$$W = \int_V (b_iu_i + c_i\theta_i)dV + \int_\Omega (t_iu_i + s_i\theta_i)d\Omega \tag{7}$$

where b_i, c_i, t_i, and s_i are the body force, body couple, traction and surface couple, respectively. Hence, the total potential energy of the deformable body using Eqs. (1) and (7) is written as

$$\Pi = U - W = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})dV - \int_V (b_iu_i + c_i\theta_i)dV - \int_\Omega (t_iu_i + s_i\theta_i)d\Omega \tag{8}$$

and its first variation gives

$$\delta\Pi = \int_V (\sigma_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ij})dV - \int_V (b_i\delta u_i + c_i\delta\theta_i)dV - \int_\Omega (t_i\delta u_i + s_i\delta\theta_i)d\Omega \tag{9}$$

3. Governing equation and pertinent boundary conditions of micro-plates

Consider an initially flat thin elastic plate of thickness h consisting of homogeneous linearly elastic material occupying the two-dimensional domain Ω of arbitrary shape in the x, y plane bounded by the curve Γ which may be piecewise smooth, i.e. it may have a finite number of corners (see Fig. 1). The plate is bending under the combined action of the distributed transverse load $q(x, y)$, the edge moment $\widetilde{M}_{nn}$ and the edge force $\widetilde{V}_n$ producing a three dimensional deformation state including the transverse deflection $w(x, y)$ and the in plane displacements $u_\alpha(x, y, z)$ which in the absence of in plane forces are written as (Timoshenko and Woinowsky-Krieger, 1959)

$$u_\alpha(x, y, z) = -zw_{,\alpha} \tag{10}$$

Taking into account Eqs. (10), (4) and that

$$w_{,1} \equiv w_{,x}, \quad w_{,2} \equiv w_{,y} \tag{11}$$

the displacement and rotation vectors of the micro-plate become, respectively,

$$\mathbf{u} = -zw_{,x}\,\mathbf{e}_1 - zw_{,y}\,\mathbf{e}_2 + w\mathbf{e}_3 \tag{12}$$

$$\boldsymbol{\theta} = w_{,y}\,\mathbf{e}_1 - w_{,x}\,\mathbf{e}_2 \tag{13}$$

Substituting Eqs. (12) and (13) into Eqs. (2) and (3) the nonzero components of the strain and curvature tensor are written as

$$\varepsilon_x \equiv \varepsilon_{11} = -zw_{,xx}, \quad \varepsilon_y \equiv \varepsilon_{22} = -zw_{,yy}, \quad \gamma_{xy} \equiv 2\varepsilon_{12} = -2zw_{,xy} \tag{14a, b, c}$$

$$\chi_x \equiv \chi_{11} = w_{,xy}, \quad \chi_y \equiv \chi_{22} = -w_{,xy}, \quad \chi_{xy} \equiv \chi_{12} = \frac{1}{2}(w_{,yy} - w_{,xx}) \tag{15a, b, c}$$

respectively.

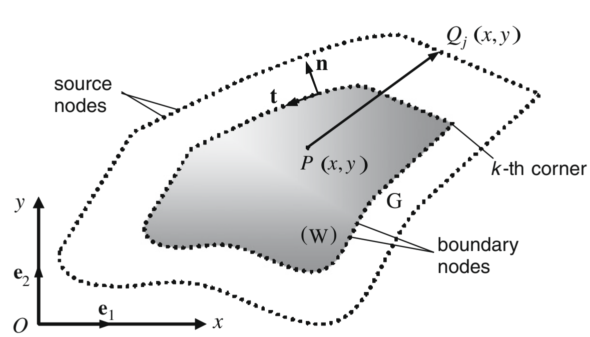

Fig. 1. Plate geometry and distribution of the boundary and source nodes.