International Journal of Solids and Structures 46 (2009) 1240–1250

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Orthotropic elastic media having a closed form expression of the Green tensor

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ARTICLE INFO

Article history:
Received 5 August 2008
Received in revised form 16 October 2008
Available online 12 November 2008

Keywords:
Elasticity
Orthotropic
Cubic
Tetragonal
Green tensor
Anisotropic

ABSTRACT

Obtaining the Green tensor for the most general orthotropic medium is not generally possible in a closed form because the solution requires the roots of a sextic, often known as Stroh eigenvalues. The paper gives some conditions under which the sextic can be solved in a closed form for any direction within the space. It enables the construction of classes of orthotropic materials for which the Green tensor can be computed in a closed form (closed-form orthotropic or CFO) for any direction within the space. The cases of transversely isotropic, tetragonal and cubic materials are studied as special cases. The comparison between the exact Green function and approximate Green functions obtained from the nearest CFO material (in the sense of four different distances) is finally performed in the case of five examples of elasticity tensors.

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1. Introduction

The Green tensor for elasticity in an infinite space is defined by the displacement field at any point within a linear elastic medium induced by a point force in any direction. It is the basis of many applications either to obtain the stress field due to defects (Mura, 1987) or to solve elasticity problems by integral equations methods. When the material is not isotropic, a fully explicit analytical solution for the Green function can be obtained for 2D problems. In the case of 3D problems a fully explicit expression of the Green tensor has been obtained only in some specific situations:

- for any direction within a transversely isotropic material (Kroner, 1953; Lejcek, 1969; Willis, 1969; Dahan and Predeleanu, 1980; Pan and Chou, 1976; Nakamura and Tanuma, 1997);
- for materials whose elastic tensors are obtained by linear transformation of the axes from transversely isotropic materials (Pouya and Zaoui, 2006; Pouya, 2007a,b) for materials characterized by the ellipsoidal anisotropy of De Saint Venant (1863);
- for orthotropic or anisotropic materials when the direction between the point where the displacement is computed and the point where the force is applied is parallel or perpendicular to some planes of symmetry (Ting and Lee, 1997; Lee, 2002).

Series solutions can also be obtained in other cases (Mura and Kinoshita, 1971; Mura, 1987; Chang and Chang, 1995; Kuznetsov, 1996; Faux and Pearson, 2000), but such solutions lead to computation times which could limit the possibility of applications. Approximate solutions can also be obtained for example in the case of cubic crystals (Dederichs and Leibfried, 1969).

For the general case of anisotropy, the solution can be put into the form of a scalar integral of a rational fraction (Lifshitz and Rozenzweig, 1947; Mura, 1987). Such a form of solution can be used for numerical purposes within the boundary element method by computing numerically the integral (Condat and Kirchner, 1987; Wang, 1997; Sales and Gray, 1998; Tonon et al., 2001; Lee, 2003). It needs, however, further developments and it induces a priori longer computation times than a closed form solution.

The main problem for obtaining a closed form of the Green tensor is that the denominator of the rational fraction which appears in the integral form of that tensor is a sixth order polynomial, whose roots cannot be obtained in a closed-form (Head, 1979) in the most general case. The purpose of the paper is to search elasticity tensors which display such a property and to show that in the case of some specific orthotropic material, the roots of the sixth order polynomial can be obtained for any direction of the space. The Green tensor can then be computed in a closed form for any direction of the space.

2. The Green tensor for an anisotropic material

The component $G_{km}(\mathbf{x}-\mathbf{y})$ of the Green tensor of an elastic medium is defined as the displacement component in the x_k-direction at point $\mathbf{x}$ when a unit body force in the x_m-direction is applied at point $\mathbf{y}$ in an infinitely extended media. These components comply to the equilibrium equations:

$$C_{ijks}\frac{\partial^2 G_{km}}{\partial x_j \partial x_s} + \delta_{im}\delta(\mathbf{x}-\mathbf{y}) = 0 \tag{1}$$

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doi:10.1016/j.ijsolstr.2008.10.033

where $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function, δ_{im} is the Kronecker delta and C_{ijks} are the elastic constants.

A classical derivation (Lifshitz and Rozenzweig, 1947; Mura, 1987) leads to the expression of the Green tensor $\mathbf{G}$ under the form of a contour integral:

$$\mathbf{G} = \frac{1}{8\pi^2 r}\int_C \mathbf{Q}^{-1}(\mathbf{k})\,ds(\mathbf{k}) \tag{2}$$

where r is the distance between point force and observation point.

The integrand is the inverse of the second order acoustic tensor $\mathbf{Q}$, whose components are the following:

$$Q_{ik}(\mathbf{k}) = C_{ijks}k_j k_s \tag{3}$$

The contour integral must be computed along the circle C of unit radius centered at the origin which is in the plane (P) perpendicular to the direction $\mathbf{x}-\mathbf{y}$.

If $\mathbf{y}$ is chosen at the origin, the cartesian coordinates of the unit vector in the direction $\mathbf{x}$ are given as functions of its spherical coordinates as follows:

$$(\sin(\phi)\cdot\cos(\theta), \sin(\phi)\cdot\sin(\theta), \cos(\phi))$$

Let $\mathbf{n}$ and $\mathbf{m}$ be two orthogonal unit vectors parallel to the plane (P); these two vectors can be chosen as follows:

- for $\mathbf{n}$: $(\sin(\theta), -\cos(\theta), 0)$;
- for $\mathbf{m}$: $(\cos(\phi)\cdot\cos(\theta), \cos(\phi)\cdot\sin(\theta), -\sin(\phi))$.

In the plane (P), the vector $\mathbf{k}$ can be expressed as:

$$\mathbf{k} = \cos\psi\,\mathbf{n} + \sin\psi\,\mathbf{m} = \cos\psi(\mathbf{n} + p\mathbf{m}) \tag{4}$$

where $p = tan(\psi)$

With these notations, Eq. (2) can be written as:

$$\mathbf{G} = \frac{1}{8\pi^2 r}\int_0^{2\pi} \mathbf{Q}^{-1}(\psi)\,d\psi \tag{5}$$

Let:

$$Q_{0ik} = C_{ijks}n_j n_s \qquad R_{ik} = C_{ijks}n_j m_s \qquad T_{ik} = C_{ijks}m_j m_s \tag{6}$$

The matrix $\mathbf{Q}(\psi)$ is a function of ψ which can be written:

$$\mathbf{Q}(\psi) = \mathbf{Q}_0\cos^2\psi + (\mathbf{R}+\mathbf{R}^T)\cos\psi\sin\psi + \mathbf{T}\sin^2\psi \tag{7}$$

$$= \mathbf{\Gamma}(p)\cos^2\psi \tag{8}$$

where

$$\mathbf{\Gamma}(p) = \mathbf{Q}_0 + p(\mathbf{R}+\mathbf{R}^T) + p^2\mathbf{T} \tag{9}$$

Finally, with the use of $p = \tan(\psi)$:

$$\mathbf{G} = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\mathbf{\Gamma}^{-1}(p)dp = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{\widehat{\mathbf{\Gamma}}(p)}{|\mathbf{\Gamma}|(p)}\,dp \tag{10}$$

where $\widehat{\mathbf{\Gamma}}$ and $|\mathbf{\Gamma}|$ are the adjoint and the determinant of $\mathbf{\Gamma}$. The components of $\widehat{\mathbf{\Gamma}}$ are polynomials of fourth order and $|\mathbf{\Gamma}|$ is a sixth order polynomial.

Computing the integral in (10) by residue calculus requires the poles located at the roots of the sixth order polynomial $|\mathbf{\Gamma}|$ which are all complex (Ting, 1996). If these poles are known and if these poles are distinct, the Green tensor is given by:

$$\mathbf{G} = \frac{1}{2\pi r}i\sum_{v=1}^{3}\frac{\widehat{\mathbf{\Gamma}}(p_v)}{|\mathbf{\Gamma}|'(p_v)} \tag{11}$$

where p_v are the roots of $|\mathbf{\Gamma}|$ with a positive imaginary part and $|\mathbf{\Gamma}|'(p)$ is the derivative of $|\mathbf{\Gamma}|(p)$.

Obtaining the roots of $|\mathbf{\Gamma}|(p)$ is not generally possible by using radicals as it is well known from the work of Galois (Head, 1979). In the general case, it is possible (Ting, 1996) to obtain these values by computing numerically the eigenvalues of the matrix

$$\begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q_0} & -\mathbf{R}(\mathbf{T}^T)^{-1} \end{bmatrix} \tag{12}$$

In some special cases, the equation $|\mathbf{\Gamma}|(p) = 0$ can be solved in a closed form; for example if the symmetry is such that the odd powers of p are cancelled in $\Gamma(p)$. In such a case, the equation is a third order equation in p^2. The equation $|\mathbf{\Gamma}|(p) = 0$ has, however, no closed form solution in the most general case. The aim of the following is to describe situations, in the case of orthotropic materials only, where the roots of $|\mathbf{\Gamma}|(p)$ can be obtained in a closed form for any direction of the vector $\mathbf{x}-\mathbf{y}$.

3. Principles for the factorization of the determinant of the acoustic tensor

The previous section has shown that computing the integral in (10) by residue calculus requires the computation of the poles located at the roots of the sixth order polynomial $|\mathbf{\Gamma}|(p)$. The properties of $|\mathbf{\Gamma}|(p)$ are, however, closely related to the properties of the determinant $\Delta(\mathbf{k}) = |\mathbf{Q}(\mathbf{k})|$ of the acoustic tensor $\mathbf{Q}(\mathbf{k})$.

In the following, the material will be assumed orthotropic and all C_{ijkl} are, in the axes of symmetry of the material, functions of nine constants which can be denoted c_{11}, c_{22}, c_{33}, c_{23}, c_{31}, c_{12}, c_{44}, c_{55}, c_{66} where the classical notation with two indices c_{ij} is used:

$$c_{ij} = C_{iijj} \quad \text{for} \quad i = 1,\ldots,3,\ j = 1,\ldots,3$$

$$c_{II} = C_{ijij} \quad \text{for} \quad I = 4,\ i = 2,\ j = 3 \quad \text{or} \quad I = 5,\ i = 3,\ j = 1 \quad \text{or} \quad I = 6,\ i = 1,\ j = 2$$

$$c_{IJ} = 0 \quad \text{for} \quad I \geqslant 4, J \leqslant 3 \quad \text{or} \quad J \geqslant 4,\ I \leqslant 3 \quad \text{or} \quad (I,J \geqslant 4 \quad \text{and} \quad I \neq J)$$

This change of notation assumes that the new matrix coefficients allow the computing of the components of the stress tensor from the components of $\gamma_{ij} = 2\epsilon_{ij}$. This leads to the matrix related to the acoustic tensor given in Appendix D.

If the coordinate axes are chosen along the axes of symmetry of the material, the determinant $\Delta(\mathbf{k})$ of the matrix related to the acoustic tensor is an homogeneous function of third order of the squares of the coordinates k_1, k_2, k_3 of $\mathbf{k}$ given by:

$$\Delta = a_{111}l_1^3 + a_{222}l_2^3 + a_{333}l_3^3 + a_{112}l_1^2 l_2 + a_{113}l_1^2 l_3 + a_{221}l_2^2 l_1 + a_{223}l_2^2 l_3 + a_{331}l_3^2 l_1 + a_{332}l_3^2 l_2 + a_1 l_1 l_2 l_3 \tag{13}$$

where $l_i = k_i^2$ and where the coefficients a_{ijk} are functions of the elastic coefficients c_{IJ} given in Appendix A.

As explained previously, the case of a transversely isotropic material (or the one of a material obtained from a transversely isotropic material by a scaling of the axes) is such that the Stroh eigenvalues can be obtained in a closed form for the 3D case, allowing the Green tensor to be obtained in a closed form.

It is easy to show that such a result is due to the fact that the determinant Δ of the acoustic tensor can be factorized by using homogeneous order 2 polynomials in k_i (or linear homogeneous polynomials in l_i).

Indeed, for a transversely isotropic material, the components of the elasticity tensor can be written as functions of five elastic constants $c_{11}, c_{33}, c_{12}, c_{13}, c_{44}$, other constants being given by:

$$\begin{aligned} c_{22} &= c_{11} \\ c_{23} &= c_{13} \\ c_{55} &= c_{44} \\ c_{66} &= \frac{1}{2}(c_{11} - c_{12}) \end{aligned} \tag{14}$$

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