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Column buckling with shear deformations—A hyperelastic formulation

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ABSTRACT

Column constitutive relationships and buckling equations are derived using a consistent hyperelastic neo-Hookean formulation. It is shown that the Mandel stress tensor provides the most concise representation for stress components. The analogous definitions for uniaxial beam plane stress and plane strain for large deformations are established by examining the virtual work equations. Anticlastic transverse curvature of the beam cross-section is incorporated when plane stress or thick beam dimensions are assumed. Column buckling equations which allow for shear and axial deformations are derived using the positive definiteness of the second order work. The buckling equations agree with the equation derived by Haringx and are extended to incorporate anticlastic transverse curvature which is important for low slenderness, high buckling modes and with increasing width to thickness ratio. The work in this paper does not support the existence of a shear buckling mode for straight prismatic columns made of an isotropic material.

1. Introduction

The correct mechanical model for the inclusion of shear deformations in the analysis of column buckling has been debated by various researchers for many years. The inclusion of shear deformations in the analysis of column buckling is very important for the design of helical springs, elastomeric bearings, sandwich plates and built-up and laced columns (see Attard, 2003; Bazant, 2003; Bazant and Beghini, 2004; Bazant and Beghini, 2006; Engesser, 1889; Engesser, 1891; Gjelsvik, 1991; Haringx, 1948; Haringx, 1949; Haringx, 1942; Kardomateas and Dancila, 1997; Reissner, 1972; Reissner, 1982; Simo et al., 1984; Simo and Kelly, 1984; Timoshenko and Gere, 1963; Zielger, 1982). Shear deformations during buckling are also important in the analysis of the compressive strength of fiber composites where fiber microbuckling models have been postulated (see Budiansky and Fleck, 1994; Fleck and Sridhar, 2002). The first to modify the Euler column buckling formula to include shear deformations was Engesser (1889),Engesser (1891). Engesser's formula for the critical buckling $P_{cr,Eng}$ of a prismatic straight column is very simple and is often written in the form:

$$\frac{1}{P_{cr,Eng}} = \frac{1}{P_{euler}} + \frac{1}{P_S} \Rightarrow \frac{P_{cr,Eng}}{P_S} = \frac{\frac{P_{euler}}{P_S}}{1 + \frac{P_{euler}}{P_S}} \tag{1}$$

where P_{euler} is the Euler buckling load and $P_S = GA$ is a so-called "shear buckling load" (G is the shear modulus and A the cross-sectional area). This shear buckling load is equivalent to Rosen (1965) microbuckling shear buckling equation, although in the microbuckling case, shear buckling is the limit taken for very large buckling wavelength. As a column's slenderness is reduced, the Engesser's critical buckling load has an upper limit of the shear buckling load GA. A critical

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question in this paper is whether shear buckling is predicted for prismatic straight columns made of an isotropic material?

Engesser's solution P_{cr_Eng} is rewritten in Eq. (2) in which l is the effective length, $r = \sqrt{\frac{I_{zz}}{A}}$ the radius of gyration, $I_{zz} = \iint_A y^2 \mathrm{d}A$ is the second moment of area and E the elastic modulus. Haringx (1942) developed an alternate buckling formula (P_{cr_Har} given in Eq. (2)) which unlike Engesser formula, predicted an infinite buckling load as the slenderness approached zero

$$P_{cr\_Eng} = \frac{\pi^2 EA}{\left(\frac{l^2}{r^2} + \pi^2 \frac{E}{G}\right)}, \quad P_{cr\_Har} = \frac{GA}{2}\left(\sqrt{1 + \frac{4P_{euler}}{GA}} - 1\right) \tag{2}$$

Haringx's formula was also adopted for helical springs. Assuming that the pitch of a compressed helical spring is insignificant, a single coil is modeled by circular rings connected by rigid bars at the center of the rings. The spring is thus replaced by an equivalent prismatic rod of suitable equivalent rigidities (the simplest approximation is to use an equivalent Poisson's ratio of $\nu = -0.8$ and $l/r = 0.94l/R_o$ where R_o is the radius of the spring helix). As well, to take account of axial shortening the following substitutions were made (see Timoshenko and Gere (1963)):

$$l = l_o\left(1 - \frac{P_{cr}}{EA_o}\right), \quad A = A_o\left(1 - \frac{P_{cr}}{EA_o}\right), \quad I_{zz} = I_{zzo}\left(1 - \frac{P_{cr}}{EA_o}\right) \tag{3}$$

where l_o, A_o and I_{zzo} are the original (before deformation) length, area and second moment of area of the spring, respectively. Allowing for shear and axial deformation the buckling formula derived in Attard (2003), Timoshenko and Gere (1963) and Goto et al. (1990) is given below and is here called the modified Haringx formula:

$$\frac{P_{cr}}{EA} = \frac{1}{2\left(1 - \frac{E}{G}\right)}\left(1 \pm \sqrt{1 - \frac{4P_{euler}}{EA}\left(1 - \frac{E}{G}\right)}\right) \tag{4}$$

Haringx's formula and the modified Haringx formula agreed well with the experimental results for short rubber rods and helical springs, respectively as shown in Figs. 1 and 2. For the helical springs the experimental results showed that for small slenderness, springs do not buckle below a slenderness of about 4.9. This observation agreed with the modified Haringx's formula but not with Engesser's which predicted buckling for any slenderness. As can be seen in these figures, Engesser's formula did not match the experimental results. It should be noted that both Engesser's and Haringx's derivations are based on a simple beam model called the Timoshenko beam. Plane sections are assumed but are not necessarily perpendicular to the centroidal axis of the beam.

There have been several authors who have discussed the merits of the differing approaches of Engesser and Haringx (see Attard, 2003; Bazant, 2003; Bazant and Beghini, 2004; Bazant and Beghini, 2006; Gjelsvik, 1991; Kardomateas and Dancila, 1997; Reissner, 1982; Simo et al., 1984; Zielger, 1982; Bazant and Cedolin, 1991). The arguments for and against were debated by Zielger (1982) who supported Engesser's approach, and Reissner (1982) who supported Haringx's approach. Zielger (1982) incorporated axial as well as shear deformations and derived the following formula which he called the modified Engesser formula:

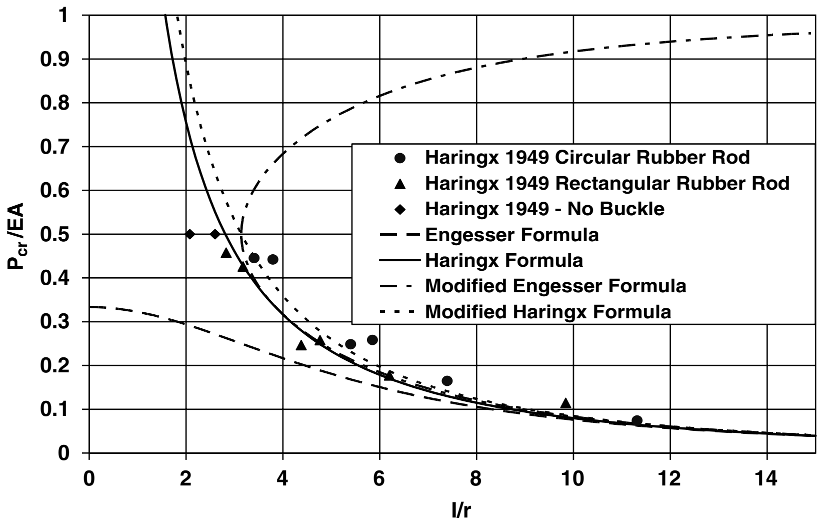

Fig. 1. Comparison of column buckling test results for rubber Rods in Haringx (1949) with predictions of Haringx and Engesser formulas.