International Journal of Solids and Structures 46 (2009) 464–479

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

A variational principle of elastoplasticity and its application to the modeling of frictional materials

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ARTICLE INFO

Article history:
Received 16 March 2008
Received in revised form 24 August 2008
Available online 30 September 2008

Keywords:
Plasticity
Friction
Geomaterials
Sand
Soil
Variational principles
Optimization

ABSTRACT

Starting from a thermomechanical description of elastoplasticity, a stress-based variational principle is derived. The principle, which generalizes von Mises's principle of maximum plastic dissipation, reproduces the conventional elastic/hardening-plastic framework applicable to metals as a special case and further proves to be suitable for developing constitutive models for frictional materials. Application of the principle to the isotropic and triaxal compression behaviour of sands is considered by means of a non-conventional extension of the modified Cam clay model. The new model allows for the specification of arbitrary stress-dilatancy relations without altering the yield potential or introducing a separate flow potential. Moreover, the elastoplastic tangent modulus is always symmetric, regardless of the degree of apparent nonassociativity.

1. Introduction

The strength and deformation characteristics of frictional materials such as clay, sands and rock are usually modeled within the framework of small-strain, rate-independent elastoplasticity. The governing equations come in the form of an additive decomposition of the total strain into elastic and plastic parts, an elastic law, a yield surface, a flow rule, and a hardening law. The benefits of casting these governing equations in terms of a variational statement have long been recognized and significant efforts have been devoted to developing variational formulations of different classes of elastoplastic models. Representative examples include Washizu (1982), Reddy and Martin (1994), Han and Reddy (2001), and Simo (1998). However, the larger part of this progress has been confined to purely cohesive materials such as metals. The generally accepted validity of associated flow theories for these materials afford the governing equations an obvious normality structure that leads naturally to a variational formulation. The principle of maximum plastic dissipation Lubliner, 1990 is a well known example. For frictional materials the situation is rather different. Here the validity of the associated flow concept is more questionable and the obstacles involved in arriving at useful variational formulations are well recognized. From a mathematical point of view, these obstacles stem essentially from the non-self-adjointness of the governing equations that results from operating with different yield and flow potentials. Thus, although some progress has been made in developing principles akin to those of associated plasticity (Hjiaj et al., 2005; Telega, 1978; Chandler, 1988; De Saxce and Bousshine, 1998), these principles are usually much weaker than the classical ones and typically require knowledge of both the stress and the deformation fields to be effective. On the contrary, classical principles require knowledge of only one set of variables (static/kinematic), after which the other set (kinematic/static) follow as the conjugate, or dual, variables.

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doi:10.1016/j.ijsolstr.2008.09.020

In this paper, the variational basis of elastoplasticity is first reexamined. This is done in the context of a general stress-based principle which derives from basic thermomechanical considerations following primarily the exposition of Simo (1998). The approach also has certain similarities to the work Collins, Houlsby, and coworkers (Collins and Houlsby, 1997; Houlsby and Puzrin, 2000) and to the 'generalized standard materials' approach of Nguyen Quoc Son and coworkers (Nguyen, 1973, 2004; Halphen and Nguyen, 1975). A detailed thermomechanical analysis is not attempted however. Indeed, emphasis is placed on the structure of the governing equations that follow from the variational principle rather than on the structure of the potential from which they derive. As such, the practical application of the principle follows what Collins et al. (2007) refer to as the 'standard extant procedure'. That is, an elastic law, a yield function, and so on are postulated a priori to eventually arrive at a model that can be calibrated against experimental data. The important point, however, is that there is not complete freedom to choose the components comprising the model. For example, it is not possible to postulate a flow potential that is completely disconnected from the yield potential. Certain rules must be obeyed. However, the structure of the governing equations are such that an appropriate degree of deviation from yield surface normality can be achieved while maintaining a normality structure of the governing equations.

The variational principle proposed comes in two different versions. The first one is quite conventional and generalizes von Mises's principle of maximum plastic dissipation. Given the strain rates, the total power of deformation is maximized subject to yield conditions cast in terms of the stresses and a set of stress-like hardening variables. The main point of interest here is that an apparent coupling between the stresses and the stress-like hardening variables follows as an integral part of the constitutive equations. This type of coupling is only rarely employed in conventional elastoplastic formulations although it has been recognized for a long time (see e.g. Collins and Houlsby (1997) and references therein). It is this coupling that allows for prescribing apparently nonassociated flow rules without introducing a separate flow potential explicitly.

Although the above mentioned coupling appears to be of significant relevance to the modeling of frictional materials, the variational principle still imposes too many restrictions on the form of the governing equations to be of practical use. Consequently, a 'relaxation' of the principle is considered. This relaxation comes about by casting the principle in incremental form. That is, instead of operating with one fixed potential, a series of potentials, each with a limited range of validity in time, are specified. This formulation is obviously somewhat weaker than classical variational formulations where the relevant potential would be a true state function of the selected state variables. The principle is, however, believed to be of considerable convenience and practical usefulness, both in terms of developing constitutive models and in terms of numerical implementation.

In the following, the theoretical aspects of the new framework are first described after which its capabilities are demonstrated by the construction of an actual constitutive model. This model is inspired by classical critical state models such as Cam clay but appears to be significantly more versatile despite requiring a similar number of material parameters. In particular, a consistent behaviour in both shear and isotropic compression can be accounted for. Furthermore, all elastoplastic tangent moduli are symmetric, regardless of the degree of apparent nonassociativity.

Standard matrix notation is used throughout the paper with bold upper and lower case letters signifying matrices and vectors respectively, and with the transpose denoted by T. Furthermore, the derivative of a function $f(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with respect to $\boldsymbol{\alpha}$ is denoted by $\nabla_\alpha f(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

2. Conventional elastoplasticity

In this section the governing equations of conventional rate-independent elastoplasticity are briefly summarized for later reference. We make use of a stress-space formulation following standard expositions (Simo, 1998; Lubliner, 1990; Chen and Han, 1988).

The fundamental characteristic of elastoplastic materials is the existence of a yield criterion that effectively limits the magnitude of the stresses:

$$F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0 \tag{1}$$

where F is the yield function, $\boldsymbol{\sigma}$ are the stresses and $\boldsymbol{\kappa}$ are a set of stress-like hardening variables that evolve according to a law that will be specified shortly.

Considering only infinitesimal deformations, a standard assumption is that the total strains can be decomposed additively according to

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{2}$$

where $\boldsymbol{\varepsilon}$ are the total strains, $\boldsymbol{\varepsilon}^e$ are the elastic strains, and $\boldsymbol{\varepsilon}^p$ are the plastic strains.

Following standard elasticity theory, it is be assumed that there exists a complementary elastic energy function, ψ^e, such that the elastic strains are related to the stresses via

$$\boldsymbol{\varepsilon}^e = \nabla_\sigma \psi^e(\boldsymbol{\sigma}) \tag{3}$$

The corresponding rate form law is given by

$$\dot{\boldsymbol{\varepsilon}}^e = \nabla^2_{\sigma\sigma} \psi^e(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} = \mathbf{C}(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \tag{4}$$