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A numerical study on the solution of the Cauchy problem in elasticity

Antonio Bilotta [a], Emilio Turco [b,*]

[a] Dipartimento di Modellistica per l'Ingegneria, Università della, Calabria, Rende (CS), Italy
[b] DAP, Università degli Studi di Sassari, Alghero (SS), Italy

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ABSTRACT

This work deals with the Cauchy problem in two-dimensional linear elasticity. The equations of the problem are discretized through a standard FEM approach and the resulting ill-conditioned discrete problem is solved within the frame of the Tikhonov approach, the choice of the required regularization parameter is accomplished through the Generalized Cross Validation criterion. On this basis a numerical experimentation has been performed and the calculated solutions have been used to highlight the sensitivity to the amount of known data, the noise always present in the data, the regularity of boundary conditions and the choice of the regularization parameter. The aim of the numerical study is to implicitly device some guidelines to be used in the solution of this kind of problems.

1. Introduction

Inverse problems in elasticity is becoming an important emerging field whose engineering applications, following Bonnet and Constantinescu (2005), can be grouped as follows:

- reconstruction of buried objects such as cracks, cavities and inclusions or residual stresses;
- identification of constitutive properties also used in model updating;
- identification of inaccessible boundary values (Cauchy problem).

The present work is devoted to the solution of Cauchy problem in linear elasticity which can be formulated as follows: given the tractions and displacements on the accessible part of the boundary of an elastic body, to evaluate the same information on the inaccessible part of the boundary. This problem has many practical applications since in actual problems sometimes the data are not complete. Moreover the same problem has to be solved, as a preliminary step, also in the identification of buried objects. In this case, following the approach presented for elastic bodies in Alessandrini et al. (2003) from a theoretical point of view and in Alessandrini et al. (2005) in a numerical contest, a complete knowledge of the traction and displacement fields along the boundary is required.

Due to the severity of the ill-posedness of the Cauchy problem, see Belgacem (2007) for example, several solution methods were proposed, as testified by the rich literature on the matter. They can approximatively be grouped as follows.

1. The quasi-reversibility method introduced in Lattès and Lions (1967) and used in the solution of the Laplace problem by using the finite difference method in Klibanov and Santosa (1991) and the mixed finite element method in Bourgeois (2005).
2. Iterative methods such as that presented in Kozlov et al. (1991) and used in the framework of the boundary element method to solve the Laplace equation, in Lesnic et al. (1997) and Jourhmane et al. (2004), Helmholtz equation, in Marin et al. (2003a), stationary Stokes system, in Bastay et al. (2006), and linear elastic problems, in Marin et al. (2001, 2002a) and Comino et al. (2007). In Lesnic et al. (1997) the problem of the convergence criterion is also considered and in order to improve the convergence rate a relaxation procedure is experimented in Jourhmane et al. (2004) and Marin et al. (2001). The conjugate gradient strategy is used in Hào and Lesnic (2000) and Bastay et al. (2001) for the Laplace equation, in Marin et al. (2002b) for elasticity, in Marin et al. (2003b) for Helmholtz-type equation and, finally, in Johansson and Lesnic (2006a) for determining the fluid velocity of a slow viscous flow. A variant of the conjugate gradient strategy, namely the Minimal Error Method, is used in Johansson and Lesnic (2006b) for the reconstruction of a stationary flow, in Marin (2009a) in the field of linear elasticity and in Marin (2009b) for Helmholtz-type equations. Finally, the Landweber–Fridman iterative method is used in Marin et al. (2004a) for Helmholtz-type equation, in Marin and Lesnic (2005) for linear elasticity and in Johansson and Lesnic (2007) for the reconstruction of a stationary flow.
3. Methods based on the minimization of an energy-like functional as proposed in Andrieux et al. (2006) for solving the Laplace equation and in Andrieux and Baranger (2008) for the analysis of elastic systems.

* Corresponding author. Tel.: +39 9720403; fax: +39 9720420.
E-mail address: emilio.turco@uniss.it (E. Turco).

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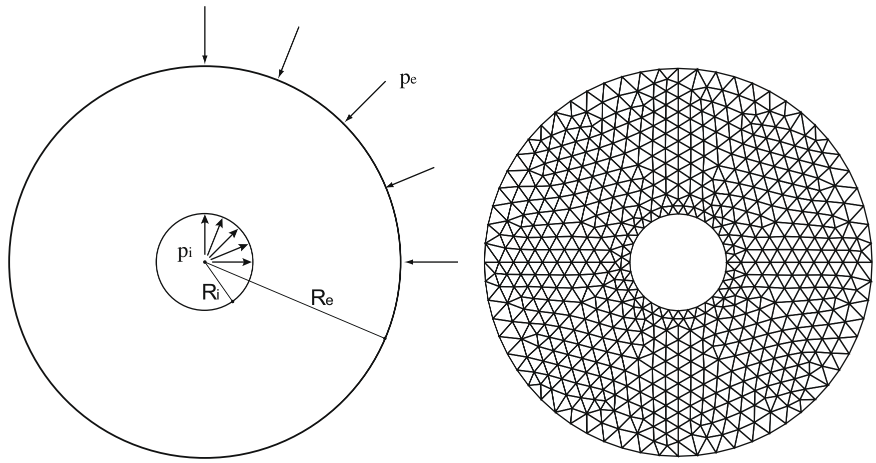

Fig. 1. Hollow cylinder under pressure test: geometry, load condition and finite element discretization.

Table 1
Meshes used for the analysis of the hollow cylinder.

Thin	*Medium*	*Thick*
$64 \times 64 \times 2$	$32 \times 48 \times 5$	$32 \times 64 \times 10$
$96 \times 96 \times 3$	$64 \times 96 \times 10$	$64 \times 128 \times 20$
$128 \times 128 \times 4$	$128 \times 192 \times 20$	$96 \times 192 \times 30$
$256 \times 256 \times 8$		

4. Methods based on the Tikhonov regularization (Tikhonov and Arsenin, 1977) have been adopted in several contributions. Among them we cite Cimetière et al. (2001) where the steady state heat equation is solved through an iterative strategy which allows to avoid the selection of the regularization parameter. In Marin and Lesnic (2002a) the elastic problem is solved by using boundary elements and the criterion used to choose the optimal solution is based on the discrepancy principle coded in the Singular Value Decomposition. In Marin and Lesnic (2002b) the optimal solution is selected by the *L*-curve criterion. Marin and Lesnic (2003), again in the analysis of the elastic problem by the boundary element method, identifies unknown portions of the boundary by using the *L*-curve method as criterion for selecting the regularization parameter and Marin et al. (2004b) presents the comparison of various regularization methods for solving problem associated with Helmholtz equation.

Finally, in order to cite also some contributions which do not fit the previous classification, in Burke et al. (2007) the identification of residual stress field in an elastic body is obtained through a partial polar decomposition, a sort of regularization similar to truncated Singular Value Decomposition, based on a spectral decomposition. Marin and Lesnic (2004), Marin (2005) and Wang et al. (2006) use the method of fundamental solutions to build a meshless strategy. Contributions (Yeih et al., 1993; Koya et al., 1993) can be framed into the so-called ficticious boundary indirect method, the first contribution proposes the theoretical approach and the second one issues some numerical details. Other interesting contributions are represented by Maniatty et al. (1989), Zabaras et al. (1989) and Schnur and Zabaras (1990). Contribution (Maniatty et al., 1989) proposes a simple diagonal regularization to determine boundary tractions using the finite element method. The concept of the spatial regularization is adopted in the frame of the boundary element method in Zabaras et al. (1989) and of the finite element method in Schnur and Zabaras (1990). Other papers use auxiliary data to recover boundary conditions. For example Maniatty and Zabaras (1994) uses the internal displacement field while Turco (1998, 1999, 2001) use the internal stress or strain fields.

The aim of the present work is to present a qualitative and quantitative study on the solution of the Cauchy problem in linear elasticity. The study is performed analyzing two-dimensional elastic problems by a standard FEM discretization. The discrete ill-posed problem derived through the discretization is tackled with the Singular Value Decomposition (Golub and Van Loan, 1996) of the problem matrix and the searched solution is calculated on the basis of a Tikhonov regularization (Tikhonov and Arsenin, 1977) of the problem.

Roughly speaking, this approach perturbs the singular values of the system matrix shifting them by an unknown parameter which plays a fundamental role in the solution of the system. The parameter, called the *regularization parameter*, makes possible to evaluate a solution of the problem and filter out the noise always present in the data. Its selection must be performed by using an external criterion, the Generalized Cross Validation (Golub et al., 1979) criterion has been adopted in the present work.

The described numerical approach has been used to study the sensitivity of the obtained solution with respect to some factors: the ratio between the length of the accessible part of the boundary and the length of the inaccessible part of the boundary, *i.e.* the amount of Cauchy data that can be used to solve the problem; the errors contained in the known boundary data; the presence of discontinuities in the domain of the problem to be solved and in the boundary conditions to be reconstructed; the chosen regularization parameter. The main aim of this numerical experimentation is to devise some guidelines useful in the solution of this kind of problems.

The paper is organized as follows. Next section describes the formulation of Cauchy problem in linear elasticity and its discretization through a standard finite element approach. Section 3 discusses the algorithm used to reconstruct the unknown boundary values. Several numerical results are presented and discussed in Section 4 and, finally, some concluding remarks, reported in Section 5, close the paper.

2. Problem keynotes

Let us consider a generic elastic body $\mathscr{B}$ loaded by the bulk force $\mathbf{f}$ and the traction $\mathbf{t}$ on the boundary $\partial\mathscr{B}$. The associated potential energy functional can be formulated as follows:

$$J := \frac{1}{2}\int_{\mathscr{B}} (\mathbf{Dd})^{\mathrm{T}}\mathbf{EDd}\,\mathrm{d}V - \int_{\mathscr{B}} \mathbf{d}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}V - \int_{\partial\mathscr{B}} \mathbf{d}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}S. \tag{2.1}$$