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Exact solution of the multi-cracked Euler–Bernoulli column

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Abstract

The use of distributions (generalized functions) is a powerful tool to treat singularities in structural mechanics and, besides providing a mathematical modelling, their capability of leading to closed form exact solutions is shown in this paper. In particular, the problem of stability of the uniform Euler–Bernoulli column in presence of multiple concentrated cracks, subjected to an axial compression load, under general boundary conditions is tackled. Concentrated cracks are modelled by means of Dirac’s delta distributions. An integration procedure of the fourth order differential governing equation, which is not allowed by the classical distribution theory, is proposed. The exact buckling mode solution of the column, as functions of four integration constants, and the corresponding exact buckling load equation for any number, position and intensity of the cracks are presented. As an example a parametric study of the multi-cracked simply supported and clamped–clamped Euler–Bernoulli columns is presented.

Keywords: Closed form solution; Euler–Bernoulli column; Stability; Concentrated cracks; Singularities; Distribution theory

1. Introduction

In many engineering problems, involving beam-like structures, continuity of the physical and geometrical properties can be interrupted by singularities due to the presence of concentrated cracks. The effect of concentrated cracks has been widely studied in the literature and models aiming at describing the variation of the flexural stiffness of the beam in the vicinity of the crack have been proposed. Without the claim of being exhaustive, some examples are recalled in what follows. In particular, a stiffness reduction due to the presence of a crack with an exponential variation law, that is not restricted to a local influence, has been proposed by Christides and Barr (1984). On the contrary, a stiffness reduction with a local effect governed by a triangular variation has been proposed by Sinha et al. (2002). Furthermore, Cerri and Vestroni (2003) proposed a constant stiffness reduction, due to a concentrated crack, limited to an effective length around the crack. Bilello (2001) modelled locally the effect of a concentrated crack by means of an ineffective area delimited by a linear reduction of its height starting from the cracked

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section. Chondros et al. (1998) modelled the crack as a continuous flexibility by using the displacement field in the vicinity of the crack, found with fracture mechanics methods. The consistent continuous model proposed by Chondros et al. (1998) is a generalization of the Christides and Barr (1984) cracked beam theory that is based on experimental determination of the exponent of the stress field. Other authors presented models of beams with transverse cracks showing that cracked structural members can be represented by a consistent static flexibility matrix (Gudmundson, 1984) and evaluating alternative expressions by including coupling terms of the flexibility influence coefficients (Okamura et al., 1969; Rice and Levy, 1972; Dimarogonas and Massouros, 1981). Papadoupolos and Dimarogonas (1987) introduced a full matrix for an arbitrary loading of a cracked beam computed by means of fracture mechanics method.

The effect of concentrated damages on the flexural stiffness in the vicinity of the crack was also treated in the literature in a macroscopic way by means of the idea of an equivalent rotational spring connecting two adjacent segments of the beam (Irwin, 1957a,b; Freund and Hermann, 1976; Gounaris and Dimarogonas, 1988; Rizos et al., 1990; Ostachowicz and Krawczuk, 1991; Paipetis and Dimarogonas, 1986). This model, based on fracture mechanics concepts, is able to capture the slope discontinuity at the cross-sections where the cracks occur. A review can be found in (Dimarogonas, 1996) with particular reference to dynamics, where a distinction between the behaviour of a notch and a crack is evidenced. In particular, a thin cut, although used to model cracks, leads to a local flexibility less than that associated to a crack.

According to the approach based on an equivalent rotational spring, the cracked beam models with continuous flexibility in the vicinity of the crack can be approximated as models with lumped flexibility by imposing that the rotation discontinuity due to the concentrated flexibility reproduces the relative rotation of the cross-sections affected by the crack.

Whatever model is adopted to describe the influence of concentrated cracks, the effects of the reduced stiffness on the deflection, on the dynamic characteristics, on the load carrying capacity, etc. should be evaluated by means of procedures able to provide accurate results with low computational effort. In particular contributions orientated towards explicit solutions are desirable for the engineering practice.

The classical method for solving problems in presence of singularities, such as concentrated cracks, relies on integration of the governing equations between singularities and on enforcement of the continuity conditions at those sections where singularities occur. Studies aiming at providing integration procedures able to treat singularities more efficiently than classical methods have been proposed in the literature to solve both static (Yavari et al., 2000, 2001a,b; Falsone, 2002) and dynamic (Dimarogonas, 1996; Gounaris and Dimarogonas, 1988; Quian et al., 1990; Morassi, 1993; Shifrin and Ruotolo, 1999) governing equations.

In particular, the study conducted in Yavari et al. (2000, 2001a,b) is very appealing since use of the distribution theory is made in the integration procedure; it provides a formulation of the governing equations over a unique integration domain however requiring, to be solved, the enforcement of a single continuity condition at each singularity.

The effects of concentrated cracks on the stability characteristics of beam structures have been also investigated in the literature by making use of the local flexibility formulation (Liebowitz et al., 1967; Liebowitz and Claus, 1968; Okamura et al., 1969; Anifantis and Dimarogonas, 1983; Takahashi, 1999; Yavari and Sarkani, 2001; Li, 2002; Fan and Zheng, 2003). Single and multiple cracks have been considered on uniform and non-uniform beams, both for Euler–Bernoulli and Timoshenko beams. Among them, procedures providing exact solutions either require enforcement of continuity conditions at each singularity or, if continuity conditions are avoided, are not able to treat all the boundary conditions. Otherwise approximate solutions able to approach functions with internal discontinuities, have been proposed.

The adoption of the distribution theory seems to be an interesting approach for problems with singularities. More precisely, besides the use of the distribution rules in the integration procedure, modelling cracks directly through distributions, such as the Dirac's delta, could lead to a robust approach.

Recently the model of flexural stiffness with singularities, represented by Dirac's deltas, has been shown to be equivalent to an internal hinge endowed with a rotational spring (Biondi and Caddemi, 2005, 2007),