International Journal of Solids and Structures 45 (2008) 5521–5539

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

An approach based on distributed dislocations and disclinations for crack problems in couple-stress elasticity

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ARTICLE INFO

Article history:
Received 27 January 2008
Received in revised form 2 May 2008
Available online 29 May 2008

Keywords:
Cracks
Dislocations
Disclinations
Couple-stress elasticity
Singular integral equations

ABSTRACT

The technique of distributed dislocations proved to be in the past an effective approach in studying crack problems within classical elasticity. The present work is intended to extend this technique in studying crack problems within couple-stress elasticity, i.e. within a theory accounting for effects of microstructure. This extension is not an obvious one since rotations and couple-stresses are involved in the theory employed to analyze the crack problems. Here, the technique is introduced to study the case of a mode I crack. Due to the nature of the boundary conditions that arise in couple-stress elasticity, the crack is modeled by a continuous distribution of climb dislocations and constrained wedge disclinations (the concept of 'constrained wedge disclination' is first introduced in the present work). These distributions create both standard stresses and couple stresses in the body. In particular, it is shown that the mode-I case is governed by a system of coupled singular integral equations with both Cauchy-type and logarithmic kernels. The numerical solution of this system shows that a cracked solid governed by couple-stress elasticity behaves in a more rigid way (having increased stiffness) as compared to a solid governed by classical elasticity. Also, the stress level at the crack-tip region is appreciably higher than the one predicted by classical elasticity.

1. Introduction

The present work introduces an approach based on distributed dislocations and disclinations (and associated singular integral equations) to deal with the mode I crack problem of couple-stress elasticity. This theory assumes that, within an elastic body, the surfaces of each material element are subjected not only to normal and tangential forces but also to moments per unit area. The latter are called couple-stresses. Such an assumption is appropriate for materials with granular or crystalline structure, where the interaction between adjacent elements may introduce internal moments. In this way, characteristic material lengths appear representing microstructure. As is well-known, the fundamental concepts of the couple-stress theory were first introduced by Voigt (1887) and the Cosserat brothers (1909), but the subject was generalized and reached maturity only in the 1960s through the works of Toupin (1962), Mindlin and Tiersten (1962), and Koiter (1964).

The theory of couple-stress elasticity assumes that: (i) each material particle has three degrees of freedom, (ii) an augmented form of the Euler–Cauchy principle with a non-vanishing couple traction prevails, and (iii) the strain-energy density depends upon both strain and the gradient of rotation. The theory is different from the Cosserat (or micropolar) theory that takes material particles with six independent degrees of freedom (three displacement components and three rotation components, the latter involving rotation of a micro-medium w.r.t. its surrounding medium). Sometimes, the name 'restricted Cosserat theory' appears in the literature for the couple-stress theory.

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doi:10.1016/j.ijsolstr.2008.05.012

It is noted that couple-stress elasticity had already in the 1960s some successful application on stress-concentration problems concerning holes and inclusions (see e.g. Mindlin, 1963; Weitsman, 1965; Bogy and Sternberg, 1967a,b; Hsu et al., 1972; Takeuti and Noda, 1973). In recent years, there is a renewed interest in couple-stress theory (and related generalized continuum theories) dealing with problems of microstructured materials. For instance, problems of dislocations, plasticity, fracture and wave propagation have been analyzed within the framework of couple-stress theory. This is due to the inability of the classical theory to predict the experimentally observed size effect and also due to the increasing demands for manufacturing devices at very small scales. Recent applications include work by, among others, Fleck et al. (1994), Vardoulakis and Sulem (1995), Huang et al. (1997, 1999), Fleck and Hutchinson (1998), Zhang et al. (1998), Anthoine (2000), Lubarda and Markenskoff (2000), Bardet and Vardoulakis (2001), Georgiadis and Velgaki (2003), Lubarda (2003), Ravi Shankar et al. (2004), Grentzelou and Georgiadis (2005), and Radi (2007).

Generally, the couple-stress theory is intended to model situations where the material is deformed in *very small* volumes, such as in the immediate vicinity of crack tips, notches, small holes and inclusions, and micrometer indentations. Examples of successful modelling of microstructure and size effects by this theory are provided by Kakunai et al. (1985) and Lakes (1995), among others. Also, a recent work by Bigoni and Drugan (2007) provides additional references and an interesting account of the determination of moduli via homogenization of heterogeneous materials.

Regarding now crack problems, there is a limited number of studies concerning such problems in couple-stress theory. Sternberg and Muki (1967) were the first to study the mode I finite-length crack elasticity problem by employing the method of dual integral equations. In their work, only asymptotic results were obtained showing that both the stress and couple-stress fields exhibit a square-root singularity, while the rotation field is bounded at the crack-tip. Adopting the same method, Ejike (1969) studied the problem of a circular (penny-shaped) crack in couple-stress elasticity. Later, Atkinson and Leppington (1977) studied the problem of a semi-infinite crack by using the Wiener–Hopf technique. More recently, Huang et al. (1997) using the method of eigenfunction expansions, provided near-tip asymptotic fields for mode I and mode II crack problems in couple-stress elasticity. Also, Huang et al. (1999) using the Wiener–Hopf technique obtained full-field solutions for semi-infinite cracks under in-plane loading in elastic–plastic materials with strain-gradient effects of the couple-stress type.

The aim of the present investigation is to extend the distributed dislocation technique (and the related singular integral equation technique) in dealing with crack problems of couple-stress elasticity and to obtain, for the first time, a *full-field* solution to the mode I problem of a *finite-length* crack. The couple-stress case is our first attempt to introduce singular integral equations in crack problems of generalized continua. Efforts dealing with gradient elasticity are also under way. Here, we introduce an approach based on distributed dislocations and disclinations. In particular, the concept of a special type of disclination (we call it 'constrained wedge disclination') is employed in order to deal with the features of the couple-stress theory. No such concept was needed in dealing with crack problems within the classical elasticity theory. For the latter problems, the standard distributed dislocation technique (DDT) was introduced by Bilby et al. (1963, 1968). This is an analytical/numerical technique. The strength of the DDT lies in the fact that it gives detailed full-field solutions for crack problems at the expense of relatively little analytical and computational demands as compared to the elaborate analytical method of dual integral equations or the standard numerical methods of Finite and Boundary Elements. A thorough exposition of the technique and the treatment of various crack problems can be found in the treatise by Hills et al. (1996).

Despite the numerous applications of the DDT in classical elasticity, it appears that there is a limited work in solving crack problems with this technique in materials with microstructure. Recently, the present authors (Gourgiotis and Georgiadis, 2007) applied the standard DDT to solve finite-length crack problems, under mode II and mode III conditions, within the framework of couple-stress elasticity. Within this framework, and having solved now the mode I (opening mode) case, a comparison between the two plane-strain crack modes (mode I and mode II) shows that mode I is mathematically more involved than mode II. Certainly, this is in contrast with situations of classical elasticity where solving problems of mode I and mode II involves the same mathematical effort. The additional effort in dealing with the mode I case here is due to the nature of the boundary conditions that arise in couple-stress elasticity (involving rotations and couple-stresses). However, such a situation does not appear in the mode II case of couple-stress elasticity (Gourgiotis and Georgiadis, 2007).

As in analogous situations of classical elasticity, a superposition scheme will be followed. Thus, the solution to the basic problem (body with a traction-free crack under a remote constant tension) will be obtained by the superposition of the stress and couple-stress fields arising in an un-cracked body (of the same geometry) to the 'corrective' stresses and couple-stresses induced by a distribution of defects chosen so that the crack-faces become traction-free. Due to the nature of the boundary conditions, it will be shown that in order to obtain the corrective solution, we need to distribute not only climb dislocations but also constant discontinuities of the rotation along the crack faces. We name the latter discontinuities *constrained wedge disclinations*. The term 'constrained' refers to the requirement of *zero* normal displacement along the disclination plane. Notice that according to the standard notion of a wedge disclination (see e.g. Anthony, 1970; de Wit, 1973), the normal displacement is also discontinuous along the disclination plane and increases linearly with distance from the core becoming unbounded at infinity. Clearly, a standard wedge disclination would not serve our purpose here. The concept of a constrained wedge disclination is first introduced in the present work (see Sections 4 and 5 below for the details).

The Green's functions of our problem (i.e. the stress fields due to a discrete climb dislocation and a discrete constrained wedge disclination) are obtained by the use of Fourier transforms. Finally, it is shown that the continuous distribution of the discontinuities along the crack faces results in a system of coupled singular integral equations with both Cauchy-type and logarithmic kernels. The numerical solution of this system shows that a cracked solid governed by couple-stress elasticity