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Equilibrium shapes of inflated inextensible membranes

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ABSTRACT

This paper proposes an effective method for directly determining the final equilibrium shapes of closed inextensible membranes subjected to internal pressures. With reference to new high-performance textile materials, we assume that the mechanical response of a fabric membrane can be accurately represented by regarding it as a two-state material. In the active state, the membrane is subject to tensile stresses and is virtually inextensible; *vice versa*, in the passive state it is unable to sustain any compressive stress, so it contracts freely. Equilibrium of the membrane in the final configuration is enforced by recourse to the minimum total potential energy principle. The Lagrange multipliers method is used to solve the minimum problem by accounting for the aforesaid nonlinear constitutive law. The set of governing equations is solved for the unknown coordinates of the equilibrium surface points. Closed form solutions are obtained for fully wrinkled cylindrical and axisymmetric membranes under homogeneous boundary conditions, while a simple iterative procedure is used to numerically solve cases of axisymmetric membranes under various inhomogeneous boundary conditions. The soundness of the proposed method is verified by comparing the results with solutions available in the literature.

1. Introduction

In addition to their traditional use in roofing structures for inexpensive coverage of large underlying areas, light, resistant and aesthetically attractive structural membranes have acquired a prominent position in many important fields of contemporary architecture and civil and industrial engineering. Here, we focus our attention on two application fields that are presently undergoing rapid growth: *deployable structures*, in which such membranes often constitute the most important component, and, *inflatable membranes*, a class of fully removable structures in which the membrane itself represents the only structural component.

Present-day structural membranes are very thin textile foils, made of two or more arrays of nearly inextensible polyester or glass filaments, which are embedded within one or more protective layers of a softer material, usually PVC or PTFE (more details can be found, for instance, in Foster and Mollaert, 2004). Such arrangements render these membranes extremely flexible, so that, from a purely structural point of view, their mechanical response may appear quite extraordinary. In fact, while on the one hand they are practically incapable of sustaining any appreciable compressive stress, on the other, only barely perceptible stretching occurs when they are subjected to tensile stresses. Because of this peculiar behaviour, studying their equilibrium states still remains a highly challenging task due to the complexity and interdependence of the geometrically nonlinear equilibrium problems involved.

By restricting the following considerations to the subject of inflatable membranes, the major source of nonlinearity undoubtedly lies in the large distances that usually separate the initial deflated configuration – in which the membrane

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is in a stress-free state – from the final one, when the membrane is fully stressed, so that the analysis must necessarily take into account large displacements/rotations. A second non-negligible source of nonlinearity is the direct consequence of the vanishing values of the membrane thickness. Although at first sight such characteristic should simplify assessment of the stress distribution, since a simpler membrane stress state prevails almost everywhere, in real membranes this is not the case. In fact, at equilibrium, the surface of an inflated membrane is made up of an unpredictable arrangement of taut and wrinkled zones, whose location is unknown *a priori*. In this regard, two aspects need to be emphasized. Stein and Hedgepeth (1961) shown that the locations of the wrinkled zones are quite different from those of the regions where the principal stresses have opposite signs, according to the predictions of standard membrane theory. Moreover, the macroscopic mechanical properties of the membrane within a wrinkled zone vary widely with the number, orientation and amplitude of the wrinkles, as first noticed by Reissner (1938), who coined the expression "*material anisotropy induced by wrinkling*". Therefore, any accurate analytical or numerical determination of the global response of the membrane cannot disregard the effects of such forms of local instability.

If we suppose that the membrane's wall still possesses some small, but non-negligible, bending stiffness, some solution can be numerically obtained by means of laborious incremental-iterative procedures based on thin shell theory. However, for decreasing thicknesses or, equivalently, increasing tensile stress levels, both the formation and evolution of wrinkling patterns on the equilibrium surface cannot be easily reproduced even by means of more refined numerical techniques, since the wavelength of the wrinkles quickly becomes smaller than the element size. On the other hand, due to the emergence of membrane locking phenomenon, further refinements of the mesh within a wrinkled region cannot improve the accuracy of results (Tessler et al., 2005).

In this paper, we present a modified membrane theory, in which the aforesaid local geometrical nonlinearities (buckling and wrinkling) are considered by means of equivalent physical nonlinearities. In particular, we assume that the membrane material is inextensible under tension but, at the same time, completely unable to resist compressive stresses. These simplifying hypotheses have many relevant consequences. Because of the assumption of vanishing thickness, the bending term of the strain energy is zero. Similarly, due to the orthogonality between the principal stresses and fictive principal stretches inherent in the assumed material constitutive law, the membrane term also vanishes. Hence, the membrane reaches its final inflated configuration following a *purely kinematic* branch, which constitutes the fundamental branch of its equilibrium path, within which large displacements/rotations occur without any internal force. This process stops at the intersection point with a secondary static branch, where an equilibrated tension-only stress state suddenly takes place within the membrane. Hereafter, increasing values of the pressure produce proportional increments in the magnitude of the stresses without any further change in the configuration.

To determine the unknown final shape, equilibrium of the membrane is enforced via the stationary (minimum) total potential energy principle (Section 2). An expressly developed application of the Lagrange multipliers method enables taking into account the aforesaid two-state constitutive law (Section 3). Closed form solutions are obtained for both cylindrical (Section 4) and axisymmetric (Section 5) membranes with homogeneous boundary conditions, while a more general iterative procedure is used to numerically solve some more complex cases of inhomogeneous boundary conditions. Homogeneous solutions refer to situations in which the membrane may inflate freely, while inhomogeneous solutions consider practical situations when a large variety of constraints are imposed on the equilibrium surface. When considering the homogeneous case of axisymmetric membranes, the "*pseudo-deformed surfaces*" first introduced by Wu (1974b), using different means, are naturally recovered.

2. Formulation of the problem

2.1. The mechanical model

Herein, the membrane in question is modeled as an ideal, perfectly flexible, two-dimensional body made of an inextensible material. In the initial configuration C_0, assumed as reference, the membrane is unloaded and stress-free, and coincides with the boundary Γ_0 of the closed region $\Omega_0 \in R^3$ (Fig. 1). When a uniform pressure p acts internally, the membrane reaches the deformed configuration C, coinciding with the unknown boundary Γ of the closed region $\Omega \in R^3$.

Over the years, a number of equilibrium problems regarding partly or fully wrinkled elastic membranes have been solved either by means of a kinematical approach (Wu, 1974a; Liu et al., 2001; Roddeman et al., 1987a,b; Barsotti et al., 2001) or, more frequently, following a physical approach (Reissner, 1938; Stein and Hedgepeth, 1961; Pipkin, 1986; Steigmann, 1990; Haseganu and Steigmann, 1994). The models following the latter approach regard wrinkles as local weakenings of the material introduced by suitably adjusting the constitutive law: the material, actually homogeneous and isotropic, is instead generally considered to be inhomogeneous and anisotropic. Here, in conformity with the aforementioned studies, we extend the physical approach to the limit case of an inextensible material.

In order to account for any possible form of local instability present in the final equilibrium shapes, in place of the real material, we substitute a new, two-state fictive one. When this material is in the active state, any surface elongation is inhibited, and a state of tensile stress arises. Contrariwise, when the fictive material is in the passive state, any contraction may freely take place, unaccompanied by compressive stresses (Barsotti and Ligarò, 2005). This is probably the simplest way to effectively represent the mechanical behaviour of modern fabric membranes.