Available online at www.sciencedirect.com
ScienceDirect
International Journal of Solids and Structures 45 (2008) 3507–3520
INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES
www.elsevier.com/locate/ijsolstr

Strain gradient theory in orthogonal curvilinear coordinates

Jidong Zhao *, Dorival Pedroso

Centre for Geotechnical and Materials Modelling, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia

Received 11 September 2007; received in revised form 11 February 2008
Available online 21 February 2008

Abstract

In this short note, general formulations of the Toupin–Mindlin strain gradient theory in orthogonal curvilinear coordinate systems are derived, and are then specified for the cases of cylindrical coordinates and spherical coordinates. Expressions convenient for practical use are presented for the corresponding equilibrium equations, boundary conditions, and the physical components for strains and strain gradients in the two coordinate systems. The results obtained in this paper are general and complete, and can be useful for a wide range of applications, such as asymptotic crack tip field analysis, cylindrical and spherical cavity expansion problems, and the interpretation of micro/nano indentation tests and bending/twisting tests on small scales.

Keywords: Strain gradient theory; Orthogonal curvilinear coordinates; Cylindrical coordinates; Spherical coordinates

1. Introduction

In early 1960s, Toupin (1962) and Mindlin (1964) proposed a strain gradient theory in which the strain energy function is assumed to depend on both the strain and strain gradient. Numerous early extensions of this theory have since been developed and used for various applications (see, e.g., Toupin, 1964; Mindlin, 1965; Mindlin and Eshel, 1968; Bleustein, 1966; Bleustein, 1967; Eringen, 1968; Eshel and Rosenfeld, 1970, 1975; Germain, 1973). The past two decades witness a revived interest in this theory from the broad community of mechanics and material science. Based on this theory, a variety of new models have been developed to investigate such problems as strain localisation and size effects in materials and challenging issues on the micro/nano scales. Conventional continuum theories fail to handle these problems due to the lack of intrinsic length scales that represent the measures of microstructure in their constitutive relations (see, e.g., Fleck and Hutchinson, 1993, 1997, 2001; Chambon et al., 1996, 1998, 2001, 2004; Georgiadis et al., 2000; Georgiadis and Grentzelou, 2006; Zhao et al., 2005, 2006, 2007a,b; Zhao and Sheng, 2006, and references cited therein). Therefore, being one of the most complete linear generalised continuum theories as commented by Georgiadis et al. (2000), the Toupin–Mindlin strain gradient theory has evidently enjoyed great success so far, and will be

* Corresponding author. Tel.: +61 2 4921 5741; fax: +61 2 4921 6991.
E-mail address: Jidong.zhao@newcastle.edu.au (J. Zhao).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2008.02.011

chosen for the study in this paper. Meanwhile, it is noteworthy that there are many other gradient theories that have also received much attention from many engineering fields. Amongst them, the gradient plasticity theory pioneered by Aifantis and co-workers (Aifantis, 1984; Zbib and Aifantis, 1988; Vardoulakis and Aifantis, 1991) is one of the most widely used. The Aifantis theory considers the Laplacian of plastic strain or other internal variables in the consistency conditions and/or flow rule, which marks its key difference with the Toupin–Mindlin theory. In this note, however, we do not attempt to make a comprehensive comparison among the various gradient theories, for which purpose the readers are referred to more recent papers such as that by Chambon et al. (2004).

The original Toupin–Mindlin Strain Gradient Theory (abbreviated hereafter as SGT) and most models based on it have been formulated in general tensor forms, which, in theory, can be recast to any specific formulations if necessary. In dealing with applications where rectangular cartesian coordinates are appropriate, one may find it straightforward and trivial to obtain the specific formulations in terms of rectangular coordinates. However, when strain gradient theories are to be used in cases where curvilinear coordinates are suitable, the corresponding formulations regarding the equilibrium equations and boundary conditions can not be obtained automatically, and the course of derivation is always exceedingly complicated yet tedious and painful. Meanwhile, formulations of strain gradient theories under orthogonal curvilinear coordinates such as cylindrical or spherical coordinates are particularly useful for a wide range of applications, such as the analysis of crack-tip field, cylindrical and spherical cavity expansion in solids, and simulation and interpretation of experiments on the microscale, such as the twisting of thin copper wires and the micro-indentation tests on various metallic materials (see Fleck et al., 1994; Nix and Gao, 1998). Limited results are available in the literature in this regard, and are mostly application-specified and thus of restricted use. For example, Bleustein (1966) have derived formulations of the SGT in spherical coordinates in a study of the stress concentration at a spherical cavity. His results, however, are confined to the axi-symmetric case. Eshel and Rosenfeld (1970, 1975) have obtained formulations of the SGT for the cylindrical tube and cavity problems, but their discussions are limited to the plane strain case only. The formulations used by Chen et al. (1999) in an investigation of the asymptotic crack-tip field by strain gradient plasticity theory apply to plane strain case only. In a recent study of the torsional surface waves in a half space, following the approach of tensor analysis outlined in Malvern (1969), Georgiadis et al. (2000) have obtained formulations of the SGT with micro inertia in terms of cylindrical coordinates. The results, however, remain limited to the special case where only one component of displacements (u_θ therein) exists. While practical problems are often complex such that simplifications are not always achievable, it is highly desirable to have a set of general formulations of SGT in terms of curvilinear coordinates that are *general and complete* enough to cover most cases and may therefore lend great convenience of immediate use for future use. To the authors' knowledge, however, such formulations are still absent, and will thus be pursued in this note.

In view of the popularity of the Toupin–Mindlin strain gradient theory as discussed above, general formulations for this theory in orthogonal curvilinear coordinates will be derived, and will then be specified for two typical systems—cylindrical coordinates and spherical coordinates. It will be demonstrated that results in many existing studies can be covered as special cases by our formulations. In the subsequent derivation, the approach and the notation used by Eringen (1967) for the translation of conventional elasticity theories from rectangular coordinates to orthogonal curvilinear coordinates are closely followed. Wherever necessary, detailed explanations will be given on uncommonly used symbols and operations. To facilitate easy comprehension, the notation used in the paper is summarized as follows:

u_i	Displacement (components)
ε_{ij}	Strain tensor
η_{ijk}	Strain-gradient tensor
σ_{ij}	Cauchy stress tensor
τ_{ijk}	Higher-order stress tensor
λ,μ	Lamé constants
ξ_i	Elastic constants associated with gradient terms
l	Internal material length scale
T_k	Surface tractions