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Buckling of thick cylindrical shells under external pressure: A new analytical expression for the critical load and comparison with elasticity solutions

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ABSTRACT

In this paper a set of stability equations for thick cylindrical shells is derived and solved analytically. The set is obtained by integration of the differential stability equations across the thickness of the shell. The effects of transverse shear and the non-linear variation of the stresses and displacements are accounted for with the aid of the higher order shell theory proposed by [Voyiadjis, G.Z. and Shi, G., 1991, A refined two-dimensional theory for thick cylindrical shells, International Journal of Solids and Structures, 27(3), 261–282.]. For a thick shell under external hydrostatic pressure, the stability equations are solved analytically and yield an improved expression for the buckling load. Reference solutions are also obtained by solving numerically the differential stability equations. Both the full set that contains strains and rotations as well as the simplified set that contains rotations only were solved numerically. The relative magnitude of shear strain and rotation was examined and the effect of thickness was quantified. Differences between the benchmark solutions and the analytic expressions based on the refined theory and the classical shell theory are analysed and discussed. It is shown that the new analytic expression provides significantly improved predictions compared to the formula based on thin shell theory.

1. Introduction

Thick shells are widely used in various engineering applications such as cooling towers, arch dams, pressure vessels etc. Parts of the human body as well can be thought of as moderately thick shells carrying fluid, for example aorta, lung airways etc. The classical theory of thin shells was developed by Love and is based on the Kirchhoff-Love assumption for the deformation in the circumferential and radial direction but it ignores radial stress effects and the transverse shear deformation.

Based on the classical thin shell theory a simple expression for the buckling load under external pressure for two dimensional isotropic shells in plain strain can be easily derived. This expression is $p_{cr}^{tsh} = \frac{1}{4}\frac{E}{1-\nu^2}\frac{h^3}{a^3}$ where a is the radius of the mid-surface of the shell, h is the thickness, E the modulus of elasticity and ν is the Poisson ratio (Timoshenko and Gere (1961)). In this expression as well as the ones that follow in the next sections, the superscript "tsh" denotes "thin shell theory". However, this expression overestimates the critical load for thick shells, i.e., leads to non-conservative results. For example, for the ratio of external to internal radius $R_2/R_1 = 0.4$ the overestimation is equal to 23.7% (Kardomateas (1993)). The reason for this overprediction can be traced to the different features that thick shells have in comparison to thin shells. For example, transverse shear can no longer be neglected while the circumferential and radial stresses vary non-linearly across the thick-

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ness of the shell. For sandwich cylindrical shells that contain a low-modulus core, the effect of transverse shear can be very dramatic (Kardomateas and Simitses (2005)).

Many theories have been developed in order to account for the effect of shear deformation. A historical review of these theories as primarily applied to the buckling problem is presented by Simitses (1996). In the first order shear deformation theory, the displacement field is assumed to vary linearly with respect to thickness (measured from the midsurface) and the rotations of the normal to the midsurface are independent variables. Fu and Waas (1995) applied this theory to study the initial post-buckling behaviour of thick rings under uniform, external hydrostatic pressure. Takano (2008) used the same theory in order to extend Flügge's (1960) stability equations for moderately thick anisotropic cylindrical shells under axial loading. Higher order shear deformation theories in which the displacements fields are expressed as cubic functions of the thickness coordinate and the transverse displacement is assumed to be constant through the thickness have been proposed by Reddy and Liu (1985) as well as Simitses and Anastasiadis (1992). This approximation results in a parabolic distribution of the transverse shear strain across the thickness. Shen (2001) employed the theory of Reddy and Liu (1985) to study the post-buckling of shear deformable cross-ply laminated cylindrical shells under combined external pressure and axial compression. Shariat and Eslami (2007) studied the buckling of thick plates using a third order shear deformation theory and obtained closed form solutions for the critical mechanical and thermal loads. The theory of Simitses and Anastasiadis (1992) was used by the same authors (Anastasiadis and Simitses (1993)) as well as Simitses et al. (1993) for the linear buckling analysis of finite- and infinite-long laminated shells under the action of external pressure. The general conclusion from these studies is that first order shear deformation theories improve significantly the predictions of the buckling load compared to the thin shell Kirchhoff-Love assumption. However, the improvement offered by the higher order theories over the first order ones is much smaller.

Voyiadjis and Shi (1991) also proposed a refined shell theory suitable for thick cylinders that incorporates not only the effect of transverse shear but also that of transverse strain and the non-linearity of the in-plane stresses. It differs from the previous theories in that the deformations in the circumferential and radial direction are obtained by solving analytically the ordinary differential equations obtained from the stress–strain relations and keeping only the low order terms in the Taylor series expansions of $\ln(z + a)$ and $1/(z + a)$. Incorporation of transverse shear deformation follows the work of Reissner (1945). The theory was first developed and applied to the problem of wave propagation in isotropic elastic plates by Voyiadjis and Baluch (1981) and was later extended for thick spherical shells by Voyiadjis and Woelke (2004).

All the above shell theories that are used to derive improved approximations of the buckling loads are based on assumptions on the distribution of the displacement field across the thickness of the shell. However, it is possible to obtain the critical loads exactly (i.e., without making such assumptions) by solving directly the stability equations of Novozhilov (1953). In this case, the displacement field is obtained as part of the solution. Kardomateas (1993), Kardomateas and Chung (1994) as well as Kardomateas and Simitses (2005) followed this approach and derived the buckling load for thick shells under external hydrostatic pressure. These exact solutions can then be used to check the accuracy of the developed shear deformation theories presented above.

The theory of Voyiadjis and Shi (1991) is used in the present paper for the estimation of the stress and moment resultants and the derivation of an improved analytical expression for the estimation of the buckling load for thick isotropic shells under external hydrostatic pressure. The accuracy of the derived expression is assessed against benchmark results obtained from the numerical solution of the differential stability equations of Novozhilov (1953).

For thin shells, strains are negligible compared to rotations so the differential stability equations contain the effect of rotations only. However, the validity of this assumption needs to be examined carefully in the context of thick shells. Kardomateas (2000) examined the effect of strains and found that they result in further decrease of the critical load.

The fact that strains are important for thick shells means that conjugate stress–strain pairs should be used to arrive at accurate benchmark results as demonstrated originally by Bažant (1971). The wider context as well as more details about different measures of finite strain and stress is provided in the book of Bažant and Cedolin (2003). It is possible to change form one conjugate pair to another using a transformation formula for the stiffness tensor. By the way, this is how the controversy between Engesser and Harinx formulae for the critical load in a beam with shear deformation is resolved. In the present paper we use the Green strain tensor and the 2nd Piola–Kirchhoff stress tensor pair. In order to assess the effect of thickness on the relative magnitude of the strains with respect to rotations, solutions were obtained using the full set (that contains rotations and strains) as well as the simpler set that contains rotations only. In order to simplify the algebra, only the latter set was used in order to derive the analytic solution. Of course, the analytic results are assessed against the numerical solution of the full set.

The paper is organised as follows: in Section 2, the differential stability equations are presented along with the associated boundary conditions for the load case under examination (hydrostatic pressure). The various steps that lead to the stability equations that contain rotations only are examined and the underlying assumptions are highlighted. This section also includes details for the numerical solution of the differential equations. In Section 3, the simplified differential equations that contain rotations only are integrated across the thickness of the shell. In the following section, the stress and moment resultants obtained from the refined shell theory are substituted to the stability equations and the resulting homogenous system is solved analytically in Section 5 yielding an improved expression for the critical pressure. Section 6 presents a detailed comparison between the derived expression and benchmark solutions for a range of h/a values. Finally, Section 7 summarises the main findings of the paper.