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Stability, bifurcation, and softening in discrete systems: A conceptual approach for granular materials

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Abstract

Matrix stiffness expressions are derived for the particle movements in an assembly of rigid granules having compliant contacts. The derivations include stiffness terms that arise from the particle shapes at their contacts. These geometric stiffness terms may become significant during granular failure. The geometric stiffness must be added to the mechanical stiffnesses of the contacts to produce the complete stiffness. With frictional contacts, this stiffness expression is incrementally nonlinear, having multiple loading branches. To aid the study of material behavior, a modified stiffness is derived for isolated granular clusters that are considered detached from the rest of a granular body. Criteria are presented for bifurcation, instability, and softening of such isolated and discrete granular sub-regions. Examples show that instability and softening can result entirely from the geometric terms in the matrix stiffness.

Keywords: Granular media; Micromechanics; Stiffness; Stability; Bifurcation; Softening

1. Introduction

The paper concerns the material behavior of granular media and examines questions of internal stability, solution uniqueness, and softening in these materials. Granular materials can be viewed as systems of granules that interact at their points of contact. The incremental boundary value problem for a granular system would involve an entire multi-grain body and the prescribed increments (rates) of displacements and external forces (Fig. 1a). When viewed as a system of nodes, connections, and supports, the problem resembles conventional problems in structural mechanics. In an alternative approach, we could treat the body as a continuum and investigate uniqueness and stability by evaluating the material behavior of the entire body or of a representative continuum point in the manner of Hill (1958), Rice (1976), and others. We suggest that questions of granular behavior can be investigated by accepting these materials as discrete systems, with the intent of appraising their susceptibility to instability and softening. The developments in the paper can be applied to the problem

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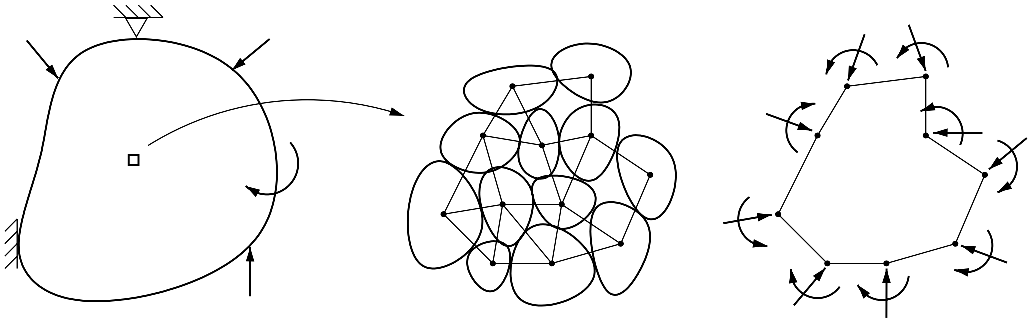

Fig. 1. Region and sub-region of a granular material: (a) granular body, (b) granular sub-region or cluster and (c) forces **b** and moments **w** on the cluster.

of an entire body and its supports, but the derivations are primarily directed toward the problem of *material behavior* within the body, perhaps the behavior within isolated sub-regions or representative volume elements (Fig. 1b). In either case, the continuum notions of stress and deformation are replaced by discrete contact forces and particle displacements within the body or sub-region (Fig. 1c). The purpose of this work is to derive the incremental stiffness of a system of particles—a stiffness that accounts for the particle shapes—and to provide stability, uniqueness, and softening criteria for the system.

In Section 2, we derive the incremental stiffness matrix for a group of N particles. The primary contribution of this section is the inclusion of geometric terms in the derivation, which account for the shapes of the particles at their contacts. By including these terms, we show that the incremental stiffness of a granular material depends, in part, on the current forces among the particles and not merely on the contact stiffnesses alone. The section includes an analysis of possible rigid rotations of a sub-region, when the sub-region is considered detached from the rest of a granular body. Section 2 ends with the recounting of a sample, prototype contact model that can be used in typical implementations. In Section 3, we present conditions for stability, uniqueness, and softening of a granular sub-region, with particular attention to the incrementally nonlinear behavior of contacts within the sub-region. Section 4 presents examples of two-particle and four-particle systems, and we end by discussing the implications of this work and possible future directions. A list of notation is given in Appendix A, and some derivations are placed in Appendices B, C, D.

2. Stiffness of a granular region

We consider the incremental motions and stiffness of an assembly or cluster of particles (Fig. 1b). Particle positions, contact forces, and loading history are assumed known at the current time t, insofar as they affect the current incremental contact stiffnesses. We address the incremental (or rate) problem in which certain infinitesimal particle motions and external force increments are prescribed, and we seek the remaining, unknown motion and force increments. The particles are assumed to be smooth and durable, with no particle breaking, and particles interact solely at their contacts (i.e., no long-range inter-particle forces). The particles are also assumed to be rigid except at their compliant contacts, where the traction between a pair of particles is treated as a point force that depends on the relative motions of the two particles. For example, this assumption would be consistent with Hertz-type contact models in which changes in force are produced by the relative approach of two particles. This compliant contact viewpoint differs, however, from "hard contact" models that enforce unilateral force and displacement constraints (Moreau, 2004). Finally, we assume slow deformations and rate-independent contact behavior.

With these assumptions, particle motions are governed by the mechanics of rigid bodies with compliant contacts: particle motions produce contact deformations; contact deformations produce contact forces; and the forces on each particle must be in equilibrium. In this section, we derive the stiffness equation for a three-dimensional group (or cluster) of N particles in the form