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Explicit expressions of $\boldsymbol{S}(v)$, $\boldsymbol{H}(v)$ and $\boldsymbol{L}(v)$ for anisotropic elastic materials

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Abstract

The three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$, appearing frequently in the investigations of the two-dimensional steady state motions of elastic solids, are expressed explicitly in terms of the elastic stiffness for general anisotropic materials. The special cases of monoclinic materials with a plane of symmetry at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$ are all deduced. Results for orthotropic materials appearing in the literature may be recovered from the present explicit expressions.

Keywords: Stroh formalism; Steady state motion; Anisotropic elasticity; Monoclinic materials; Orthotropic materials

1. Introduction

It is known that Stroh formalism is mathematically elegant and technically powerful in determining the two-dimensional deformations not only in *anisotropic elastostatics* (Stroh, 1958; Ting, 1996) but also in *anisotropic elastodynamics* (Ting, 1996). In elastostatics, the three real matrices $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$, called *Barnett–Lothe tensors* appear often in the solutions of many anisotropic boundary value problems. Due to their importance, the explicit expressions of Barnett–Lothe tensors have been investigated by many researchers. The most general anisotropic materials without any symmetry plane assumed were considered by Wei and Ting (1994) and Ting (1997). Other related works for anisotropic elastic materials with special symmetry plane assumed are Dongye and Ting (1989) for orthotropic materials, Ting (1992) and Suo (1990) for monoclinic materials with the symmetry plane at $x_3 = 0$, Tanuma (1996), and Nakamura and Tanuma (1996) for transversely isotropic materials.

In elastodynamic problems, if the elastic body is in a steady state motion in a certain direction with a constant speed $v > 0$, then the solutions of these problems are often related to three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$.

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Explicit expressions of these three matrices are also useful in the derivation of the secular equation when surface wave speed in an anisotropic elastic half-space is concerned (Ting, 2002). These matrices reduce, respectively, to *Barnett–Lothe tensors* $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$ when $v = 0$. For orthotropic materials, Dongye and Ting (1989) presented the explicit expressions of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$. Chadwick and Wilson (1992a,b) formulate these tensors in terms of integrals for monoclinic materials with the symmetry plane at $x_3 = 0$. Explicit expressions are then deduced for the special case of orthotropic and cubic materials.

In this paper, we obtain the explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for general anisotropic materials. The approach developed in our recent work for elastostatic problems (Liou and Sung, submitted for publication) is extended to the present derivations for the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$. In previous work (Liou and Sung, submitted for publication), the Barnett–Lothe tensors $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$ were expressed in terms of elastic stiffness for general anisotropic materials. Similarly, all elements of matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are also expressed in terms of elastic stiffness and results for elastosatics (Liou and Sung, submitted for publication) may be recovered when $v = 0$. Explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ derived for general anisotropic materials are then specialized to the cases of monoclinic materials with the plane of symmetry at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$. In particular, the results for the monoclinic materials with symmetry plane at $x_3 = 0$ remain valid for the degenerate cases when repeated eigenvalues occur. Moreover, our results of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for orthotropic materials may recover to those previously presented by Dongye and Ting (1989).

Below is the plan of our work. In Section 2, the Stroh formalism for the steady state motions of anisotropic elastic solids is outlined. In Section 3 the approach used by Liou and Sung (submitted for publication) was extended to the constructions of the eigenvectors for the steady state problems for anisotropic materials. With eigenvectors constructed in Section 3, the explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are then derived in Section 4. In Sections 5–7, results for materials with symmetry plane at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$ are deduced. In Section 8, our results were validated by the special case of orthotropic materials which were obtained by Dongye and Ting (1989) and finally in Section 9 we conclude our work.

2. The Stroh formalism

Consider a linear elastic body in a steady state motion in the x_1-direction with a constant speed $v > 0$. The governing equation for the displacement $\boldsymbol{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ for the two-dimensional deformations for which u_i $(i = 1, 2, 3)$ are independent of x_3 is

$$(\boldsymbol{Q} - \rho v^2 \boldsymbol{I}) \frac{\partial^2 \boldsymbol{u}}{\partial x_1 \partial x_1} + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) \frac{\partial^2 \boldsymbol{u}}{\partial x_1 \partial x_2} + \boldsymbol{T} \frac{\partial^2 \boldsymbol{u}}{\partial x_2 \partial x_2} = \boldsymbol{0}, \tag{2.1}$$

where ρ is the mass density, $\boldsymbol{I}$ is a 3×3 unit real matrix, the superscript T stands for the transpose and

$$\boldsymbol{Q} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} c_{16} & c_{12} & c_{14} \\ c_{66} & c_{26} & c_{46} \\ c_{56} & c_{25} & c_{45} \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} c_{66} & c_{26} & c_{46} \\ c_{26} & c_{22} & c_{24} \\ c_{46} & c_{24} & c_{44} \end{bmatrix}. \tag{2.2}$$

Here the contracted notations of the elastic stiffness c_{ijks} are used to express all the elements of $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{T}$ as shown above. Note that both $\boldsymbol{Q}$ and $\boldsymbol{T}$ are symmetric and positive definite. In what follows, only the subsonic problems are considered (Ting, 1996). Therefore, the general solution to Eq. (2.1) can be expressed as follows:

$$\boldsymbol{u} = 2\,\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\}, \tag{2.3}$$

where

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3], \tag{2.4}$$

$$\boldsymbol{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3)]^{\mathrm{T}}, \tag{2.5}$$

and $z_k = x_1 - vt + p_k x_2$. Unknown complex number p_k and constant vector $\boldsymbol{a}_k$ are determined by the eigenrelation