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On the generalized Barnett–Lothe tensors for monoclinic piezoelectric materials

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Abstract

The three *generalized Barnett–Lothe tensors* $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$, appearing frequently in the investigations of the two-dimensional deformations of anisotropic piezoelectric materials, may be expressed in terms of the material constants. In this paper, the eigenvalues and eigenvectors for monoclinic piezoelectric materials of class m, with the symmetry plane at $x_3 = 0$ are constructed based on the extended Stroh formalism. Then the three generalized Barnett–Lothe tensors are calculated from these eigenvectors and are expressed explicitly in terms of the elastic stiffness instead of the reduced elastic compliance. The special case of transversely isotropic piezoelectric materials is also presented.

Keywords: Stroh formalism; Generalized Barnett–Lothe tensors; Monoclinic piezoelectricity; Transversely isotropic piezoelectricity; Anisotropic solid

1. Introduction

It is known that Stroh formalism is mathematically elegant and technically powerful in determining the two-dimensional deformations of *anisotropic elastic solids* (Stroh, 1958; Ting, 1996). The three real 3×3 matrices $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$, called Barnett–Lothe tensors appear often in the solutions of many anisotropic linear elastic boundary value problems. Due to its importance, the explicit expressions of Barnett–Lothe tensors have been investigated by many researchers for purely elasticity problem. The most general anisotropic materials without any symmetry plane assumed were considered by Wei and Ting (1994) and Ting (1997). Other related works for anisotropic elastic materials with special symmetry plane considered are cited in the paper, e.g., by Soh et al. (2001). However, all their representation of the three real matrices $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$ are essentially in terms of the reduced elastic compliances.

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The extension of the Stroh formalism to the investigation of *anisotropic piezoelectricity* was made very earlier by Barnett and Lothe (1975). In recent years, to better understanding the intrinsic coupling effect between mechanical and electrical field, this extended Stroh formalism is widely used in the piezoelectric research field. In applying this extended formalism to anisotropic piezoelectricity problem, similar to the purely elasticity problem, solutions may often be expressed in terms of three real 4×4 matrices $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$, or now called *generalized Barnett–Lothe tensors* (Ting, 1996). As noted by Soh et al. (2001), based on the extended Stroh formalism, the eigenvalues and the associated eigenvectors are not able to be expressed explicitly for piezoelectric materials which make the explicit expressions of the generalized Barnett–Lothe tensors unavailable. To circumvent this, the modified Lekhinitskii formalism was adopted by them (Soh et al., 2001) to construct the eigenvalues and the corresponding eigenvectors for the general anisotropic piezoelectric materials and the explicit expressions of the generalized Barnett–Lothe tensors for transversely isotropic piezoelectric materials are presented. Their results are all in terms of the reduced elastic compliances. The approach of constructing the eigenvalues and the corresponding eigenvectors from the Lekhinitskii formalism was earlier used by many researchers e.g. Suo (1990), Ting (1992) for purely elastic materials.

In this paper, we construct the eigenvalues and eigenvectors for monoclinic piezoelectric materials of class m, with the symmetry plane at $x_3 = 0$ based on the extended Stroh formalism in a straightforward manner. The eigenvalues and eigenvectors are directly related to the elastic stiffness instead of the reduced elastic compliance. With algebraic calculations (Wei and Ting, 1994), the explicit expressions of the three generalized Barnett–Lothe tensors are presented in a concise form. Results for the special case of transversely isotropic piezoelectric materials are also given. Below is the plan of our work. In Section 2, the extended Stroh formalism is outlined. Then in Section 3 the eigenvalues and eigenvectors are constructed for monoclinic piezoelectric materials with the symmetry plane at $x_3 = 0$. In Section 4 the generalized Barnett–Lothe tensors are computed. In Section 5, the special case of transversely isotropic piezoelectric materials is presented, and finally in Section 6 concludes the work.

2. The extended Stroh formalism

In this section, the extended Stroh formalism is introduced in the following. The convention that all Latin indices range from 1 to 3 (except where indicated) and repeated indices imply summation are all followed. Bold-faced symbol stands for either column vectors or matrices depending on whether lower-case or upper-case is used. In a rectangular coordinate system x_i $(i = 1, 2, 3)$, it is known that for two-dimensional piezoelectric materials the generalized displacement vector $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^{\mathrm{T}}$ (u_i, $i = 1, 2, 3$: the elastic displacements; u_4: electric potential) and the generalized stress function vector, $\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]^{\mathrm{T}}$ are expressed as

$$\boldsymbol{u} = 2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\}, \tag{2.1}$$

$$\boldsymbol{\varphi} = 2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)\}, \tag{2.2}$$

where

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4], \tag{2.3}$$

$$\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4], \tag{2.4}$$

$$\boldsymbol{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^{\mathrm{T}}, \tag{2.5}$$

where the superscript T appearing above stands for the transpose and $z_k = x_1 + p_k x_2$. Unknown complex number p_k and constant vector $\boldsymbol{a}_k$ are determined by the eigenrelation

$$\boldsymbol{U}\boldsymbol{a}_k = \boldsymbol{0} \quad (k = 1, 2, 3, 4), \tag{2.6}$$

where

$$\boldsymbol{U} = [\boldsymbol{Q} + p_k(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p_k^2 \boldsymbol{T}], \tag{2.7}$$

$$\boldsymbol{Q} = \begin{bmatrix} c_{i1k1} & e_{1i1} \\ e_{1k1}^T & -\alpha_{11} \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} c_{i1k2} & e_{1i2} \\ e_{2k1}^T & -\alpha_{12} \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} c_{i2k2} & e_{2i2} \\ e_{2k2}^T & -\alpha_{22} \end{bmatrix}. \tag{2.8}$$