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The Eshelby stress tensor, angular momentum tensor and scaling flux in micropolar elasticity

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Abstract

The (static) energy-momentum tensor, angular momentum tensor and scaling flux vector of micropolar elasticity are derived within the framework of Noether's theorem on variational principles. Certain balance (or broken conservation) laws of broken translational, rotational and dilatational symmetries are found including inhomogeneities, elastic anisotropy, body forces, body couples and dislocations and disclinations present. The non-conserved *J*-, *L*- and *M*-integrals of micropolar elasticity are derived and discussed. We gave explicit formulae for the configurational forces, moments and work terms.

Keywords: Eshelby stress tensor; Micropolar elasticity; Conservation integrals

1. Introduction

Symmetries and conservation laws of micropolar elasticity are of important interest in mathematical physics, material science and engineering science. Jarić (1978, 1986) and Dai (1986) studied conservation laws in micropolar elastostatics. Vukobrat (1989) obtained some conservation laws for micropolar elastodynamics. The Noether theorem was applied by Pucci and Saccomandi (1990), Nikitin and Zubov (1998), Maugin (1998), Lubarda and Markenscoff (2003) to obtain conservation laws and the corresponding conserved currents for linear micropolar elasticity. For couple stress elasticity the Noether theorem was used by Lubarda and Markenscoff (2000). All these investigations were restricted to homogeneous, source-free and compatible micropolar elasticity. The derived conservation laws correspond to variational invariance of the strain energy with respect to translation and rotation symmetries. Therefore, the *J*- and *L*-integrals are conserved in such a version of micropolar elasticity. On the other hand, the scaling symmetry is not a variational symmetry in micropolar elasticity, thus, the *M*-integral is not conserved (see, e.g., Lubarda and Markenscoff, 2003).

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Not so many results on conservation laws are known for micropolar elasticity with defects. Only the Eshelby stress tensor and the configurational forces caused by defects are known (Kluge, 1969a,b). The Eshelby stress tensor corresponds to translation symmetry and it may be identified with the (static) energy-momentum tensor (EMT). But nothing is known in the literature for non-homogeneous micropolar elasticity with body forces, body couples, dislocations and disclinations present.

It is the purpose of the present paper to extend the results of these earlier results on micropolar elasticity to account for material non-homogeneity, anisotropy, defects, body forces and body couples. We will derive balance laws breaking the translation, rotation and scaling symmetries. The symmetry breaking terms are called configurational forces, configurational moments and configurational work. In turn, we find the expressions for the Eshelby stress tensor, angular momentum tensor and scaling flux in micropolar elasticity.

2. Basic equations of micropolar elasticity

In this section, we recall the basics of micropolar elasticity (Eringen, 1999). We consider the general case of anisotropic linear micropolar elasticity theory for non-homogeneous and incompatible media with defects. The strain energy for a micropolar material reads

$$W = \int w \, \mathrm{d}V \tag{2.1}$$

with the energy density

$$w = \frac{1}{2} A_{ijkl}\gamma_{ij}\gamma_{kl} + B_{ijkl}\gamma_{ij}\kappa_{kl} + \frac{1}{2} C_{ijkl}\kappa_{ij}\kappa_{kl}, \tag{2.2}$$

where γ_{ij} denotes the elastic micropolar distortion tensor and κ_{ij} is the elastic wryness tensor. These elastic strain tensors are given in terms of a displacement vector u_i and a microrotation ϕ_i. Additionally, the total strains may be decomposed into elastic and plastic parts according to

$$\gamma^{\mathrm{T}}_{ij} = \partial_j u_i + \epsilon_{ijk}\phi_k = \gamma_{ij} + \gamma^{\mathrm{P}}_{ij}, \tag{2.3}$$

$$\kappa^{\mathrm{T}}_{ij} = \partial_j \phi_i = \kappa_{ij} + \kappa^{\mathrm{P}}_{ij}. \tag{2.4}$$

Here γ^{P}_{ij} is the plastic distortion and κ^{P}_{ij} is the plastic wryness. For simplicity, we have assumed a linear relationship but that is not at all necessary. The constitutive relations for full anisotropy read:

$$t_{ij} = \frac{\partial w}{\partial \gamma_{ij}} = A_{ijkl}\gamma_{kl} + B_{ijkl}\kappa_{kl}, \tag{2.5}$$

$$m_{ij} = \frac{\partial w}{\partial \kappa_{ij}} = B_{klij}\gamma_{kl} + C_{ijkl}\kappa_{kl}, \tag{2.6}$$

where A_{ijkl}, B_{ijkl} and C_{ijkl} are the elastic tensors of micropolar elasticity with the symmetries

$$A_{ijkl} = A_{klij}, \qquad C_{ijkl} = C_{klij}. \tag{2.7}$$

Dimensionally, $[C_{ijkl}] = \ell[B_{ijkl}] = \ell^2[A_{ijkl}]$, where ℓ is a material length parameter. For the non-homogeneous medium under consideration, they depend on position, $A_{ijkl}(x)$, $B_{ijkl}(x)$ and $C_{ijkl}(x)$. t_{ij} is the force stress tensor and m_{ij} is the couple stress tensor. For an isotropic micropolar material the elastic tensors simplify to

$$\begin{aligned} A_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu_c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\ C_{ijkl} &= \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\ B_{ijkl} &= 0, \end{aligned} \tag{2.8}$$

where μ is the shear modulus, λ denotes the Lamé constant, and μ_c, α, β and γ are additional material constants for micropolar elasticity.

The field equations in presence of an external force f_i and an external couple l_i are given by