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Analysis of second-order shear-deformable beams with semi-rigid connections

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ABSTRACT

In this paper a new beam finite element is proposed for the solution of framed structures with semi-rigid connections. The element uses the second order Timoshenko's beam model. Power series expansion of kinematic relations is employed to improve solution stability. Effect of flexible and eccentric connections is considered by means of rotational and linear springs plus rigid end-offset. The proposed approach has the same computational cost of a *standard* beam element since additional degrees of freedom are condensed out. The resulting element stiffness matrix expression is given as the summation of the second order Timohenko's beam stiffness matrix and corrective matrices. The proposed element is readily implementable in existing finite element codes. Some benchmark problems are used to test the efficiency of the proposed procedure.

1. Introduction

Steel framework structural behavior is highly variable depending by the nature of the connection joining beam to columns. Common practice and wide experimental investigations have clearly demonstrated that most of these connections are semi-rigid. Therefore the adequate corresponding idealization has to be provided by a proper semi-rigid connections describers. Several are the models developed to account the effect of such type of connections. Briefly we can collect them in *empirical/experimental* models and *analytical/numerical* models [1]. Empirical models develop mathematical representations of the frame connections by means of the geometrical and mechanical properties. These methods require a fine calibration based on experimental tests. Analytical methods develop mathematical representations of the frame connections by idealizing the connection flexibility as ruled by a finite numbers of parameters. Given the above scenario here we present a new analytical method along the line presented in [2] and then further developed in [3–6] where a computer implementable second order classical beam finite element is presented and then further enriched to account for flexible and eccentric connections. Rotational springs are used to model the semi-rigid connections and rigid end-offset are used to model the joint dimension [7–12].

Several authors have proposed models for frame structures with semi-rigid connections [3–6] lately extended addressing to inelasticity property of steel material [13].

In this paper we focus the attention on the improvement of the theory of basic beam models: a new finite element model is presented for the analysis of steel frameworks. A Timoshenko's beam, whose stiffness matrix is computed by a power expression of trigonometric functions, is provided by a rotational spring and is considered to be ended by two rigid connections. Given the above mentioned attributes the model is supposed to be suitable for the study of short beams or composite beams where the shear deformations are consistent.

In this preliminary study, a linear relationship between moment and joint rotation is assumed. Next we will consider the effect, on the proposed model, of non-linear moment-rotation relationship, that is of paramount importance dealing with seismic design. Primary purpose of the present paper is to produce a finite element model readily implementable in extant commercial codes. To easily fulfill this requirement additional degrees of freedom are condensed-out in order to obtain a beam element with the same degrees of freedom used for a rigid connection [14,4]. This approach is known to be correct if the size of the connection is limited.

1.1. Overview

The paper is organized as follows. In the first section the governing equations of the problem are illustrated. In Section 2 we develop the formulation of the structural element by investigating the effect of eccentric and flexible connections. Corrective matrices for the semi-rigid approach are developed in order to obtain the final stiffness matrix as summation of each contribution. In this section the vector of generalized forces and the mass matrix are also presented. In Section 3 some numerical examples are given to test the accuracy of the presented model in the linear dynamic range. Some conclusive remarks end the paper.

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2. Governing differential equations

Consider a plane member of length l subjected to a uniformly distributed force $q_y(x)$ acting perpendicular to the beam axis before deformation, see Fig. 1. A cartesian coordinate system (x, y) and the displacements (u, v) are defined at the initial configuration of the member. According to the Timoshenko's beam theory the second order differential equations are expressed in the following form

$$\left[EA\left(u' + \frac{1}{2}v'^2\right)\right]' = 0, \tag{1}$$

$$\left[Nv'\right]' + \left[kG_0A\left(v' + \psi\right)\right]' + q_y(x) = 0, \tag{2}$$

$$\left(EI\psi'\right)' - kG_0A\left(v' + \psi\right) = 0, \tag{3}$$

where the displacements components u and v are the centroidal axis and rotation ψ appear as primary unknowns. E is modulus of elasticity, A is the cross-sectional area and I is the moment of inertia, and $q_y(x)$ is the lateral distributed load, while superscript $(\cdot)'$ denotes partial derivative with respect to x. For simplicity the following scalar parameter k has been introduced.

$$k^2 = \frac{A}{I}\left(u' + \frac{1}{2}v'^2\right). \tag{4}$$

Eqs. (1)–(3) are non-linear and coupled, hence, an analytical solution cannot generally be obtained. The relations between internal forces and displacements are given as:

$$N = EA\left(u' + \frac{1}{2}v'^2\right), \tag{5}$$

$$S = k_0G_0A\left(v' + \psi\right), \tag{6}$$

$$M = EI\psi'. \tag{7}$$

Supposing firstly that distributed load $q_x(x)$ is not applied along the axial direction, we can consider that $N = N_1 = cost$ and in this case the Eq. (2) can be analytically solved respect to v regardless the Eq. (1) [2]. Supposing also $N_1 > 0$, and setting $q_y(x) = 0$, the solution of Eqs. (1)–(3), results to be:

$$v(x) = v_1 + \frac{S_1 l^3}{EI}\frac{1}{(kl)^3}\left[kx - \frac{\sin(kx)}{(1-\alpha k^2)}\right] + \frac{M_1 l^2}{EI}\frac{(1-\cos(kx))}{(kl)^2} - \frac{\psi_1 \sin(kx)}{k(1-\alpha k^2)}, \tag{8}$$

$$\psi(x) = \psi_1\cos(kx) - \frac{S_1 l^2}{EI}\frac{(1-\cos(kx))}{(kl)^2} - \frac{M_1 l}{EI}\frac{\left(1-\alpha k^2\right)\sin(kx)}{kl}, \tag{9}$$

$$u(x) = u_1 - \frac{N_1}{EA}x - \frac{1}{2}\int_0^x v'^2 \mathrm{d}x, \tag{10}$$

where:

$$k = \sqrt{\frac{|N_1|}{EI}}, \alpha = \frac{EI}{k_0G_0A}. \tag{11}$$

If $N_1 < 0$ the solutions are obtained by replacing $kl = jkl$ and using the relations $\sin h(kl) = -j\sin(jkl)$ and $\cos h(kl) = \cos(jkl)$. At this stage if we impose $\alpha = 0$ (no shear deformation) we recover the equations presented by Goto in [2] for a second-order elastic analysis of framed structures.

2.1. Stiffness equation expressed by power series

From the equations shown in the previous section the stiffness equations are derived upon power series expression of Eqs. (8) and (9). These new arguments for the stiffness equations guaranty the development of a computer-aided model. In fact, without this power series expression, the stiffness equations will change depending the axial force N_1 is positive or negative.

Moreover, if the axial force N_1 inclines to zero, the coefficients of the stiffness equations become indefinite, causing numerical instability. On the other hand the power series expression is not affected by the numerical instability when N_1 inclines to zero. The following development in power series for trigonometric functions is considered:

$$\left.\begin{matrix}\sin(kl)\\ \sinh(kl)\end{matrix}\right\} = kl + kl\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{N_1 l^2}{EI}\right)^n, \tag{12}$$

$$\left.\begin{matrix}\cos(kl)\\ \cosh(kl)\end{matrix}\right\} = 1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(-\frac{N_1 l^2}{EI}\right)^n. \tag{13}$$

The corresponding stiffness equations obtained from Eqs. (8)–(10) are

$$\begin{bmatrix} S_1 \\ M_1 \\ S_2 \\ M_2 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12\xi_1 & 6\xi_2 & -12\xi_1 & 6\xi_2 \\ 6\xi_2 & 4\xi_3 & -6\xi_2 & 2\xi_4 \\ -12\xi_1 & -6\xi_2 & 12\xi_1 & -6\xi_2 \\ 6\xi_2 & 2\xi_4 & -6\xi_2 & 4\xi_3 \end{bmatrix}\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2, \end{bmatrix} \tag{14}$$

where the corrective functions ξ_i have the following expressions:

$$\xi_1 = \frac{1}{12\xi}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\left(-\tilde{N}_1\right)^n\right]\left(1-\alpha k^2\right), \tag{15}$$

$$\xi_2 = \frac{1}{6\xi\tilde{N}_1}\left[-\sum_{n=1}^{\infty}\frac{1}{(2n)!}\left(-\tilde{N}_1\right)^n\right], \tag{16}$$

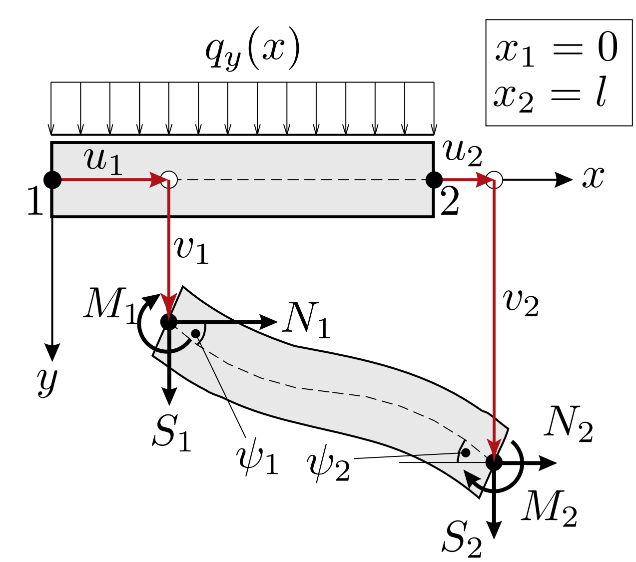

Fig. 1. *Coordinate system for member.*