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# Tunnelling effects for acoustic waves in slowly varying axisymmetric flow ducts

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ABSTRACT

The multiple-scales Wentzel–Kramers–Brillouin (WKB) approximation is used to model the propagation of acoustic waves in an axisymmetric duct with a constriction in the presence of mean flow. An analysis of the reflection/transmission process of modes tunnelling through the constriction is conducted, and the key mathematical feature is the presence of two turning points, located at either real axial locations or in the complex plane. The resulting asymptotic solution consists of WKB solutions in regions away from the constriction and an inner solution valid in the near vicinity of the constriction. A solution which is uniformly valid throughout the duct is also derived. A range of test cases are considered, and the importance of accounting for the inner region, even in cases in which the turning points lie away from the real axis, is demonstrated.



## 1. Introduction

The interest in reducing noise emission from machinery such as aircraft engines has resulted in a considerable effort directed towards studying sound transmission through flow ducts. In many cases these ducts are axially non-uniform, but their properties vary relatively slowly with distance along the duct, and this allows asymptotic analysis to be used to derive simplified solutions. In this direction a key advance was made by Rienstra [1], who considered a circular lined duct with a mean flow with the duct radius varying slowly along the axis. Since then a number of extensions have been made, including to ducts of arbitrary cross section with mean flow [2], ducts with a mean swirling flow [3], curved ducts [4] and ducts carrying axially sheared vortical mean flow [5]. In each of these papers formulae for both the local axial wave numbers and the axial variation of the modal amplitudes are derived using the Wentzel–Kramers–Brillouin (WKB) approximation.

The WKB analysis of modal propagation along a slowly varying duct breaks down and becomes singular when the mode undergoes a transition from cut-on to cut-off, at a so-called turning point. Several of the above-mentioned references consider the case of a single turning point, and show that an incident wave is fully reflected with a phase shift $-\pi/2$, so that only an evanescent wave is transmitted beyond the turning point. In this case the solution in an inner region around the turning point is governed by Airy's equation. Multiple turning points are possible, however, and a typical scenario might involve a duct with a constriction in which, over a particular frequency range, a cut-on incident mode becomes cut-off as it approaches the constriction but then becomes cut-on again once the constriction is passed (i.e. two turning points). If these two turning points are close together then the flow in the inner region containing the turning points is governed by the

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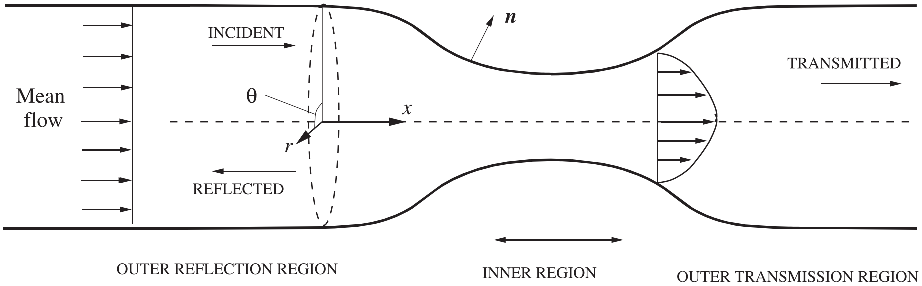


**Fig. 1.** Illustration of the duct with a constriction. Sound is incident from the left.

parabolic cylinder equation, and the incident wave 'tunnels' through the inner region and is partially transmitted. This situation is entirely analogous to the solution of Schrödinger's equation for scattering by a parabolic potential. This latter problem was considered by Keller [6], who derived both expressions for the reflection and transmission coefficients and a uniformly valid approximation which is valid in the vicinity of, and far away from, the turning points.

Ovenden [7] has found a uniformly valid solution, in the style of [6], for modes undergoing a cut-on to cut-off transition at a single turning point in a slowly varying duct with mean flow. However, the case of two nearby turning points, both in terms of determining the reflection and transmission coefficients and in terms of deriving a uniformly valid solution, has not been completed in the acoustic-flow case, and this is therefore the focus of the current paper. As we will see, the presence of the two turning points can have a very marked effect on the acoustic propagation, even in cases when the turning points are complex and are no longer present in the flow domain.

The paper is set out as follows. In Section 2 the equations governing the flow and acoustic field are presented. The slowly varying modes, i.e. those that are either well cut-on or those far away from the transition points, are presented in Section 3. In Section 4 the wave motion through the duct constriction is studied, and reflection and transmission coefficients are determined. A uniformly valid approximation is derived in Section 5. Specimen results are presented in Section 6. We note that the non-uniform results presented in this section 4 have previously been presented in section 3.1 of [8]. The key original result of our paper is the uniformly-valid solution, which will be given in the next section.

## 2. Problem formulation

We adopt much of the formulation and notation used by [1,7]. The duct has a hollow circular cross-section of axially varying radius, and we shall focus for definiteness on the case in which the duct has a single constriction, as indicated in Fig. 1. The duct walls are modelled as being perfectly rigid. There is a steady mean flow, which is purely axial and uniform in the parallel sections of the duct at infinity. The acoustic medium is assumed to be compressible, inviscid and isentropic. In what follows we non-dimensionalise distance by the duct radius far upstream, $R_\infty$, and speed and density by the sound speed and density in the undisturbed fluid, $c_\infty$ and $\rho_\infty$, respectively. We introduce the axial coordinate $x$ and polar coordinates $r, \theta$ in the cross-section of the duct. The radius of the duct is assumed to be slowly varying in the axial direction, and we introduce a slow scale, $X = \epsilon x$, where $\epsilon \ll 1$ is a measure of the axial slope of the duct walls. The local duct radius is denoted by $R = R(X)$. An acoustic wave propagates along the duct in the positive $X$ direction, and potentially becomes cut-off as it approaches the constriction, resulting in a wave reflected back up the duct and a transmitted wave on the far side of the constriction – see Fig. 1. We shall conduct the derivation with a positive going flow for convenience, but shall later alter the flow direction by a sign change.

The total flow field is decomposed into a mean flow plus a small time-harmonic perturbation [1],

$$[\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{p}, \tilde{c}] = [\mathbf{V}, D, P, C] + [\nabla\phi, \rho, p, c]\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1}$$

where $\tilde{\mathbf{v}}$ is the total velocity field and $\tilde{\rho}, \tilde{p}, \tilde{c}$ are the total density, pressure and sound speed. Capital letters denote mean quantities. The flow field is governed by the equations of mass and momentum conservation, together with the isentropic relationships [9]

$$\tilde{c}^2 = \tilde{\rho}^{\gamma-1} \tag{2}$$

$$\gamma\tilde{p} = \tilde{\rho}^{\gamma}, \tag{3}$$

where $\gamma$ is the ratio of specific heats. Note that in (1) the velocity perturbation is irrotational, thanks to the incident perturbation being an acoustic wave. This allows the introduction of the unsteady velocity potential, $\phi$, which satisfies the convected wave equation [1]

$$\nabla\cdot(D\nabla\phi) - D(-\mathrm{i}\omega + \mathbf{V}\cdot\nabla)\left[\frac{1}{C^2}(-\mathrm{i}\omega + \mathbf{V}\cdot\nabla)\phi\right] = 0. \tag{4}$$